

# On the Characteristics and Verification of Tenable Strategy Sets in Bimatrix Games

TYLER PORTER

I consider the robust set-valued equilibrium concepts *coarse tenability* and *fine tenability* introduced in [Myerson and Weibull 2015]. I show how their framework of *consideration-set* games maps onto the more familiar framework of strategy perturbations. This allows me to compare these set-valued concepts to other objects from the equilibrium refinement literature, as well as provide methods to verify whether strategy sets satisfy these robustness properties in bimatrix games. I provide complexity results for the verification of *fine tenability* and show how the methods developed for verifying this concept can be applied to proper equilibria.

## ACKNOWLEDGEMENTS

I thank Jörgen Weibull for assistance with understanding the block solution concepts. David Easley provided a number of helpful discussions on the direction of the paper. Eva Tardos and Larry Blume provided assistance with technical aspects and organization of the paper. I would also like to thank four anonymous referees at EC'20. This paper would not be complete without their help.

## 1 INTRODUCTION

Since the seminal works of [von Neumann and Morgenstern 1944] and [Nash 1951], game theoretic methods have been brought to bear on problems of strategic interaction of every variety. The need for solution concepts with more cutting power than Nash equilibrium has led to the vast literature on equilibrium refinements. Many of these refinements come with different levels of agent sophistication or particular game forms in mind. In this paper, I study the characteristics and provide methods for the verification of the robust set-valued equilibrium concepts introduced in [Myerson and Weibull 2015]. These concepts are motivated by, on the one hand, models of evolutionary game theory in which a population of players is called upon to engage in strategic interaction with norms and historical precedent playing a role; and, on the other hand models of rationalizable choice and inattention. A central idea is that within a population a particular convention arises in which only a certain collection of strategies for players in various type roles are ever played. A natural question then is which conventions survive the test of time? In order to answer this question, [Myerson and Weibull 2015] discuss multiple solution concepts that describe when a convention is robust against small deviations from convention within the population. This paper is part of the growing literature on equilibrium refinement computation. I do not attempt to survey the entirety of this literature here, but I will briefly discuss those works which are most related to this one.

Few papers deal with the computation or verification of set-valued equilibrium concepts. [Benisch et al. 2010] provide methods to compute the minimal sets which are closed under rational behavior (CURB) of [Basu and Weibull 1991] in polynomial time by building such sets upon individual strategies. For finite games of 3 or more players, it was shown in [Klimm et al. 2011] that the problem of computing a minimal CURB set as well as the problem of verifying whether a given set is a minimal CURB set are both NP-complete. In this same paper, it is shown that computing strong CURB sets (in which correlated strategies are allowed) can be done in polynomial time by modifying the methods of [Benisch et al. 2010]. [Brandt and Brill 2016] discuss the computation of several set-valued solution concepts which can be expressed using the dominance structures defined in [Duggan and Breton 2014]. [Jansen and Vermeulen 2001] provide a framework for verifying whether a collection of equilibria satisfy *stability* as defined by [Kohlberg and Mertens 1986]. I adapt their framework in Section 4 below for the current problem.

Much of the literature on the computation of equilibria and equilibrium refinements focuses on point-valued concepts. The problem of computing a Nash equilibrium in a bimatrix game is known to be PPAD-complete due to a series of results by [Chen and Deng 2006] and [Daskalakis et al. 2006]. [Conitzer and Sandholm 2008] extend the results of [Gilboa and Zemel 1989] to show that many problems such as approximating the maximum social welfare at any Nash equilibrium are hard even for symmetric bimatrix games. For papers discussing the complexity and computation of equilibrium refinements in finite games see [Miltersen and Sørensen 2010], [Hansen et al. 2010], [Hansen 2017] and references therein.

It is important to note that the complexity and computation of proper equilibria is particularly relevant to the discussion in this paper. The concept of *fine tenability*, formally defined later, is quite similar to proper equilibrium in that it imposes a rationality requirement on trembles. I show that the exact methods I develop for fine tenability can also be applied to the verification of proper equilibria in bimatrix games. [Hansen and Lund 2018] show that the complexity of verifying the conditions imposed by proper equilibrium is NP-complete in bimatrix games. I show a similar result for verifying the conditions of fine tenability by providing a nontrivial modification of their reduction. [Sørensen 2012] showed that the problem of finding a single proper equilibrium in a bimatrix game can be done by applying Lemke’s algorithm to a linear complementarity problem of polynomial size, and hence is PPAD-complete. The discussion in sections 5 and 6 of this paper provide an answer to the question of how small  $\epsilon$  needs to be to guarantee that an  $\epsilon$ -proper equilibrium profile is indeed close to a proper equilibrium, though this bound in general cannot be determined efficiently. This was briefly mentioned as an issue in [Sørensen 2012], although the method proposed in that paper did not require this information. [Belhaiza et al. 2012] propose ad-hoc methods for computing proper equilibria by first solving a mixed 0-1 quadratic program and analytically verifying solution candidates. They note that the informal analytical approach of iteratively satisfying the criteria of  $\epsilon$ -properness was introduced by [Myerson 1991]. This paper provides a formal generalization of this method using the language of hyperplane arrangements as it applies to the problem of verifying fine tenability. This characterization is, to the best of my knowledge, not fleshed out in any of the previous literature, and sheds light on the geometry of fine tenability and proper equilibria. This characterization allows for a clean discussion of the complexity of verifying fine tenability, and also provides explicit bounds for the  $\bar{\epsilon}$  used in the definition of fine tenability (See definition 3 below) in bimatrix games.

## 2 PRELIMINARIES

I consider finite games of complete information defined as  $G = (N, S, u)$ . The set  $N$  denotes the set of players, with  $|N| = n$ . The set  $S = \times_{i=1}^N S_i$  denotes the set of pure strategy profiles, with  $S_i$  denoting the set of pure strategies for player  $i$ . The mapping  $u_i : S \rightarrow \mathbb{R}$  assigns to every pure strategy profile  $s$  a payoff for player  $i$ . The space of mixed strategies for player  $i$  is given by  $\Delta S_i$ , the collection of probability distributions over strategies in  $S_i$ , with generic elements given by  $\sigma_i$ . The set of mixed strategy profiles is denoted  $\Sigma = \times_{i=1}^N \Delta S_i$ , with generic element  $\sigma$ . For a mixed strategy profile, the payoff to player  $i$  is given by,

$$u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s)$$

where  $\sigma(s) = \prod_{i=1}^N \sigma_i(s_i)$ . With the usual abuse of notation,  $u_i(s_i, s_{-i})$  denotes the payoff of player  $i$  from playing  $s_i \in S_i$  when all other players play according to  $s \in S$ . Similarly,  $u_i(s_i, \sigma_{-i})$  is defined as the payoff to player  $i$  when she plays pure strategy  $s_i$  while all other players randomize according to  $\sigma \in \Sigma$ .

For a given mixed strategy  $\sigma_i$ , let  $Y_i(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$  denote the *carrier* of  $\sigma_i$ . For a mixed strategy profile  $\sigma$ , let  $BR_i(\sigma) = \{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \ \forall s'_i \in S_i\}$  denote the set of best responses for player  $i$  to the profile  $\sigma$ . A profile  $\sigma$  is a *Nash Equilibrium* of  $G = (N, S, u)$  if and only if for each player  $i$ ,  $Y_i(\sigma_i) \subseteq BR_i(\sigma)$ .

In this paper, I discuss the robust strategy blocks introduced in [Myerson and Weibull 2015]. In particular, I am interested in the verification of blocks that are said to be *coarsely tenable* and *finely tenable*. I make no nondegeneracy assumptions about the finite game in question. Before proceeding to the relevant equilibrium concepts, I introduce the framework of *consideration-set games* from [Myerson and Weibull 2015].

For a set  $E$ , let  $\mathbb{P}(E)$  denote the power set of  $E$ . Let  $C_i$  denote a generic element of  $\mathbb{P}(S_i)$ . Every set  $C_i$  is referred to as a "type" for player  $i$ . If a player is of type  $C_i$ , then strategies within  $C_i$  are precisely the strategies that player  $i$  considers using. All strategies outside of  $C_i$  are played with probability 0 if player  $i$  is of type  $C_i$ . There is a probability distribution  $\mu_i$  over types for each player. That is,  $\mu_i(C_i)$  denotes the probability that player  $i$  is of type  $C_i$ . These distributions are independent for each player. The joint distribution over  $\mathbb{P}(S)$  is denoted by  $\mu$ . This defines a consideration set game  $G^\mu = (N, S, u, \mu)$ .

A pure strategy for player  $i$  is a mapping  $f_i : \mathbb{P}(S_i) \rightarrow S_i$  with the property that  $f_i(C_i) \in C_i$ . Denote the space of all pure strategies for player  $i$  by  $F_i$ . Mixed strategies are defined in the usual way, as randomizations over the pure strategy mappings. The associated space of mixed strategies for player  $i$  is denoted by  $\Delta F_i$ . A generic element of  $\Delta F_i$  is denoted by  $\tau_i$ , and a strategy profile of the consideration set game is denoted by  $\tau$ . Given  $\tau$ , let  $\tau_{i|C_i}$  denote the conditional probability distribution over pure strategies in  $C_i$  when player  $i$  is of type  $C_i$ . Each profile  $\tau$  induces a probability distribution  $\tau^\mu$  over the pure strategies of each player in the following way

$$\tau^\mu(s_i) = \sum_{\{C_i: s_i \in C_i\}} \tau_{i|C_i}(s_i) \mu_i(C_i)$$

This projection of  $\tau$  onto the space of mixed strategies allows one to compute the payoffs to each player given a profile  $\tau$ . For a profile  $\tau$ , denote the induced mixed strategy profile by  $\tau^\mu$ .

A profile  $\tau$  is said to be an equilibrium of the consideration-set game if for each player  $i$  and each  $C_i \in \mathbb{P}(S_i)$ ,

$$u_i(\tau_{i|C_i}, \tau_{-i}^\mu) = \max_{s_i \in C_i} u_i(s_i, \tau_{-i}^\mu)$$

That is to say, if player  $i$  is of type  $C_i$  then player  $i$  chooses only best responses (from among those strategies contained in  $C_i$ ) to the probability distribution induced by  $\tau_{-i}$ . The equilibria are defined ex-ante. The following definitions will refer to subsets (referred to as "blocks")  $T = \times_{i=1}^N T_i$  of the collection of pure strategy profiles. Each  $T_i \subseteq S_i$  is a nonempty collection of pure strategies for each player.

**Definition 1** ([Myerson and Weibull 2015]). A strategy block  $T \subseteq S$  is said to be *coarsely tenable* if  $\exists \bar{\epsilon} \in (0, 1)$  such that for all  $\epsilon \in (0, \bar{\epsilon})$  and each type distribution  $\mu$  such that  $\mu(T) > 1 - \epsilon$ ,

$$\max_{s_i \in S_i} u_i(s_i, \tau_{-i}^\mu) = \max_{t_i \in T_i} u_i(t_i, \tau_{-i}^\mu)$$

for player  $i$  and each equilibrium  $\tau$  of the incomplete information game induced by  $\mu$ .

*Coarse tenability* requires that the block  $T$  be robust against all sufficiently small deviations from the convention  $T$ . The notion of *fine tenability* is a weaker concept in that it requires robustness against a specific class of deviations from convention. In particular, the deviations that are said to be  $\epsilon$ -proper. The definition of  $\epsilon$ -properness for strategy blocks is quite similar to the one defined for strategy profiles defined by Myerson in his definition of *proper* equilibria (see [Myerson 1978]). Indeed, the definition imposes a requirement on the relative probability of trembles between "more"

and "less" rational alternatives of strategy blocks. Here, rationality is used to describe the number of strategic alternatives that a player considers when playing as an unconventional type.

**Definition 2.** [Myerson and Weibull 2015] Given a strategy block  $T$  and an  $\epsilon \in (0, 1)$ , a type distribution  $\mu$  is said to be  $\epsilon$ -proper if, for every player  $i$ :

- $\mu_i(T_i) > 1 - \epsilon$
- $\mu_i(C_i) > 0, \forall C_i \subseteq S_i$
- $T_i \neq C_i \subset D_i \implies \mu_i(C_i) \leq \mu_i(D_i)\epsilon$

Where  $C_i$  and  $D_i$  are arbitrary subsets of  $S_i$ .

The first point in the definition requires that players are sufficiently likely to be of a conventional type. The second imposes a requirement that every possible type has a positive probability under  $\mu$ . The third point reflects the requirement that players place greater probability on more rational deviations from convention.

I now introduce the final set-valued equilibrium concept of fine tenability.

**Definition 3.** [Myerson and Weibull 2015] A block  $T$  is *finely tenable* if  $\exists \bar{\epsilon} \in (0, 1)$  such that for all  $\epsilon \in (0, \bar{\epsilon})$  and for any type distribution  $\mu$  that is  $\epsilon$ -proper,

$$\max_{t_i \in T_i} u_i(t_i, \tau_{-i}^\mu) = \max_{s_i \in S_i} u_i(s_i, \tau_{-i}^\mu)$$

for each player  $i$  and each equilibrium  $\tau$  of the of the incomplete information game induced by  $\mu$ .

As noted by Myerson and Weibull (2015), if a block  $T$  is coarsely tenable, then it also satisfies the weaker condition of fine tenability. It is important to note that as  $\mu(T) \rightarrow 1$ , the projections of equilibria of the consideration-set games  $G^\mu$  converge to equilibria of the support restricted game  $G^T = (N, T, u)$ . This fact is given in [Myerson and Weibull 2015], and is used throughout the analysis in this paper.

### 3 CHARACTERIZATION USING PERTURBATIONS

In order to obtain a more concrete understanding of the structure of these block concepts and relate them to other solution concepts in the game theory literature, it is helpful to rephrase them in terms of the more familiar language of strategically perturbed games. A perturbation vector  $\delta_i \in \mathbb{R}_+^{|S_i|}$  assigns to every strategy  $s_i \in S_i$  a minimum probability  $\delta_i(s_i)$  that this pure strategy must receive at any mixed strategy profile. These perturbation vectors must of course satisfy  $\sum_{s_i \in S_i} \delta_i(s_i) \leq 1$ . Given a tuple of perturbations  $\{\delta_i\}_{i=1}^N$ , the set of admissible mixed strategies for player  $i$  is defined as  $\{\sigma_i \in \Delta S_i : \sigma_i(s_i) \geq \delta_i(s_i) \forall s_i \in S_i\}$ . Let  $Y_i^\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > \delta_i(s_i)\}$  denote the  $\delta$ -carrier of the mixed strategy  $\sigma$ .

**Definition 4.** An equilibrium of the perturbed game  $G = (N, S, u, \delta)$  is a mixed strategy profile  $\sigma$  such that for each player  $i$ ,  $Y_i^\delta(\sigma_i) \subseteq BR_i(\sigma)$ .

With the appropriate definitions established, I proceed with relating projections of equilibria of consideration-set games to Nash equilibria of appropriately perturbed games. Keeping in mind that the final goal is to examine the properties of tenable strategy sets, I will suppose throughout this section that the analysis is given with an arbitrary, but fixed, strategy block  $T$ .

**Lemma 1.** Fix a block  $T$ , an arbitrary distribution over types  $\mu$ , and an equilibrium of the associated consideration-set game  $\tau$ . Consider the induced probability distribution over pure strategies  $\tau^\mu$ .

*Claim:* There is a probability distribution  $\mu'$  such that the marginal probability distributions  $\mu'_i$  have the property that

- $\mu'_i(T_i) = \mu_i(T_i)$

- $\mu'_i(C_i) = 0$  for all  $C_i$  such that  $C_i \neq T_i$  and  $|C_i| \neq 1$

( $\mu'_i$  places positive probability only on  $T_i$  and individual strategies in  $S_i$ ) and there is an equilibrium  $\tau'$  of the consideration-set game induced by  $\mu'$  with  $\tau^\mu = \tau'^\mu$ . That is to say,  $\tau'$  induces the same probability distribution over pure strategies as  $\tau$ .

All proofs are in the appendix. The idea is that for a given equilibrium  $\tau$  with projection  $\tau^\mu$ , reconstruct the distribution  $\tau^\mu$  directly using type distributions over individual strategies and  $T_i$ . Distributions of the form  $\mu'$  described in the above lemma will be referred to as *simple* distributions throughout the rest of this paper. Note that when discussing *simple* distributions over types one requires the context of a specific  $T$  in order for the definition to make sense. The correct terminology would be "*simple with respect to  $T$* ". I omit this in future mentions of *simple* distributions with the understanding that the distributions are *simple* with respect to the block  $T$  in question.

Since deviations of the form described in the above lemma must be considered anyway, Lemma 1 says that there is no loss in generality in restricting attention to such simplified probability distributions. These distributions can be viewed in some sense as the familiar strategy perturbations used in describing various equilibrium concepts in the literature (*proper*, (*strictly*) *perfect*, etc.). The precise connection will soon become clear. For a mixed strategy profile  $\sigma$ , let  $BR_i^T(\sigma) = \{s_i \in T_i : u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \ \forall s'_i \in T_i\}$  denote the set of best responses among strategies in  $T_i$  for player  $i$  against profile  $\sigma$ .

**Definition 5.** For a fixed strategy block  $T$  and perturbation tuple  $\delta = \{\delta_i\}_{i=1}^N$ , a profile  $\sigma$  is a *T-equilibrium* of the perturbed game  $(N, S, u, \delta)$  if for each player  $i$ ,  $Y_i^\delta(\sigma_i) \subseteq BR_i^T(\sigma)$ .

Note that *T-equilibria* need not coincide with Nash equilibria of the perturbed game. However, these are precisely the profiles which correspond to projections of equilibria of consideration-set games as  $\mu(T) \rightarrow 1$  in a strong sense.

**Lemma 2.** For a fixed block  $T$  in the game  $G = (N, S, u)$  and for every *simple* type distribution  $\mu$  with  $\mu(T) > 0$ , there is a perturbation tuple  $\delta = \{\delta_i\}_{i=1}^N$  with  $\sum_{s_i \in S_i} \delta_i(s_i) = 1 - \mu_i(T_i)$  for each player  $i$  and such that for every equilibrium  $\tau$  of the consideration-set game  $G^\mu$ , there is a *T-equilibrium*  $\sigma$  of the perturbed game  $(N, S, u, \delta)$  such that  $\tau^\mu = \sigma$ .

A variant of the reverse direction of Lemma 2 is also true.

**Lemma 3.** For a fixed strategy block  $T$ , each perturbation tuple  $\delta = \{\delta_i\}_{i=1}^N$  and every *T-equilibrium*  $\sigma$  of the perturbed game  $G = (N, S, u, \delta)$ , there is a *simple* type distribution  $\mu$  such that  $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$  and such that there is an equilibrium  $\tau$  of the consideration-set game  $G^\mu$  in which  $\sigma = \tau^\mu$ .

Lemmas 2 and 3 provide a simple way to express the relationship between consideration-set games and strategically perturbed games. This correspondence allows for the comparison of the set-valued equilibrium concepts found in [Myerson and Weibull 2015] to other solution concepts found in the game theory literature. In particular, I show that every *coarsely tenable* strategy block  $T$  contains a set that is *stable* in the sense of [Kohlberg and Mertens 1986] (henceforth *KM-stable*). Before proceeding, I provide a definition of *KM-stability* used in [Jansen and Vermeulen 2001] for the verification of *KM-stable* sets in bimatrix games.

**Definition 6** (Jansen and Vermeulen (2001)). A closed set  $E$  of strategy pairs is called a *KM-set* if for each neighborhood  $V$  of  $E$ , there is  $\eta > 0$  such that if the strategy perturbations satisfy  $\|\delta\| < \eta$ ,

then there is an equilibrium of the perturbed game lying in  $V$ . A set  $E$  that is minimal with respect to this property is referred to as  $(KM)$ -stable.<sup>1</sup>

Note here that  $\|\delta\|$  refers to the Euclidean norm of the full tuple of perturbations  $\{\delta_i\}_{i=1}^N$  in  $\times_{i=1}^N \mathbb{R}^{|S_i|}$ .

**Proposition 1.** In a finite game  $G = (N, S, u)$ , if a strategy block  $T$  is *coarsely tenable*, then  $T$  contains a set that is  $KM$ -stable.

It is worth noting that this result is known to be true for generic finite games (see [Wikman 2016]). I will return to the relationship between consideration-set games and perturbed games when discussing the verification of coarse tenability in the next section. I now move on to characterizing the relationship between projections of equilibria of consideration-set games with  $\epsilon$ -proper type distributions and tuples  $(\sigma, \delta)$  with specific structure.

**Definition 7.** Consider a fixed strategy block  $T$  and  $\epsilon > 0$ . A tuple  $(\sigma, \delta)$  consisting of a mixed strategy profile  $\sigma$  and a strategy perturbation  $\delta$  is said to be  $T(\epsilon)$ -proper if the following conditions hold:

- $s_i \in \operatorname{argmax}_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \geq \delta_i(s_i)$
- $s_i \notin \operatorname{argmax}_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) = \delta_i(s_i)$
- $\delta_i(s_i) > 0$
- $u_i(s_i, \sigma_{-i}) > u_i(s_j, \sigma_{-i}) \Rightarrow \delta_i(s_i)\epsilon \geq \delta_i(s_j)$
- $\sum_{\{s_i \in S_i\}} \delta_i(s_i) \leq \epsilon$

Note that the profile  $\sigma$  need not be  $\epsilon$ -proper, however the perturbations  $\delta$  satisfy a properness criterion under the ordering over strategies induced by  $\sigma$ . The idea behind much of the work in the rest of this section is once again to consider all probability that arises from deviations from convention  $T$  as perturbations. The proof that for a fixed  $T$  the equilibria of consideration-set games with  $\epsilon$ -proper type distributions admit tuples which are  $T(\epsilon)$ -proper is almost exactly the same proof as Proposition 2 of [Myerson and Weibull 2015]. A proof is provided here for purposes of completeness and to note some subtle differences.

**Lemma 4.** Fix a strategy block  $T$ . Suppose that  $\mu$  is an  $\epsilon^3$ -proper type distribution and that  $\tau$  is an equilibrium of the consideration-set game  $G^\mu$ . Then there is a tuple  $(\sigma, \delta)$  that is  $T(\epsilon)$ -proper and such that  $\tau^\mu = \sigma$ .

Now, I would like to show that any tuple that is  $T(\epsilon)$ -proper corresponds to an equilibrium of a consideration-set game with an  $\epsilon$ -proper type distribution. Unfortunately, this in general requires type distributions which are proper according to  $\epsilon^{\frac{1}{z}} > \epsilon$  for a given  $T(\epsilon)$ -proper tuple  $(\sigma, \delta)$  and  $z \in \mathbb{N}$ . I will need to introduce some notation before continuing. Consider a mixed strategy profile  $\sigma$ . This profile induces a total preorder over the strategies of each player according to expected payoffs. Let  $\succsim_i$  denote the total preorder of the form  $s_i \succsim_i s_j$  if and only if  $u_i(s_i, \sigma_{-i}) \geq u_i(s_j, \sigma_{-i})$  with  $\sim$  denoting indifference. This total preorder partitions the strategy space of player  $i$  into indifference classes. Denote these indifference classes by  $\zeta_1^i, \zeta_2^i, \dots, \zeta_{r_i}^i$  with  $r_i \leq |S_i|$ . The classes are ordered such that  $s \in \zeta_k^i$  and  $s' \in \zeta_{k+1}^i$  implies  $u_i(s, \sigma_{-i}) < u_i(s', \sigma_{-i})$ .

**Proposition 2.** For a fixed strategy block  $T$ , there exists  $\epsilon^* > 0$  and  $z \in \mathbb{N}$  such that for all  $\epsilon \in (0, \epsilon^*)$  and any tuple  $(\sigma, \delta)$  that is  $T(\epsilon)$ -proper, there is an  $\epsilon^{\frac{1}{z}}$ -proper type distribution and an equilibrium  $\tau$  of the associated consideration-set game such that  $\tau^\mu = \sigma$ .

<sup>1</sup>Note that Kohlberg and Mertens (1986) required that the set  $E$  be composed only of Nash equilibria. Imposing minimality on  $E$  ensures that the sets considered in Jansen and Vermeulen (2001) are indeed the sets defined as *stable* under the definition of Kohlberg and Mertens.

**Remark 1.** The proof of this proposition is essentially an application of the following observation. Fix  $k \in \mathbb{N}$  and  $M > 0$ . There is an  $\bar{\epsilon} \in (0, 1)$  and  $z \in \mathbb{N}$  such that for any collection  $\alpha_0, \alpha_1, \dots, \alpha_k \in [0, M]$  and  $\epsilon \in (0, \bar{\epsilon})$ ,

$$\sum_{i=0}^k \frac{\alpha_i}{\epsilon^z} \leq \frac{1}{\epsilon}$$

Note that precisely how small  $\epsilon^*$  in Proposition 2 needs to be is unimportant. Since fine tenability only considers behavior under small perturbations, it suffices to show that there is an equivalence between the two concepts in a neighborhood of the game where  $T$  is played with probability 1.

#### 4 VERIFICATION OF COARSELY TENABLE BLOCKS

Given a strategy block  $T$ , a natural question is how to determine whether it is *coarsely tenable*. This section is devoted to determining how to answer this question in the case of two players. Throughout the remainder of this paper, I consider a bimatrix game in which player 1 has pure strategy set  $S_1$  and player 2 has pure strategy set  $S_2$ . I suppose that  $|S_1| = m$  and  $|S_2| = n$ . The associated mixed strategy sets are given by  $\Delta S_1$  and  $\Delta S_2$ . The set of mixed strategy profiles is denoted by  $\Sigma = \Delta S_1 \times \Delta S_2$ . A generic element of  $\Delta S_1$  is denoted by  $p$ , while a generic element of  $\Delta S_2$  is denoted by  $q$ . Given a profile  $(p, q)$ , payoffs are determined by the matrices  $A_{m \times n}$  and  $B_{m \times n}$  in the following way:

$$u_1(p, q) = pAq$$

$$u_2(p, q) = pBq$$

Nash equilibria of this game are the strategy pairs  $(p, q)$  such that

$$pAq \geq p'Aq \quad \forall p' \in \Delta S^1$$

$$pBq \geq pBq' \quad \forall q' \in \Delta S^2$$

Throughout this section, I assume that I am armed with a concise description of the Nash equilibria of the full bimatrix game. The set of all Nash equilibria in a bimatrix game is a finite union of convex polytopes using results of [Jansen 1981]. Furthermore, there exist methods of computing all Nash equilibria of a bimatrix game in finite time, regardless of assumptions on nondegeneracy (see [Avis et al. 2010]).

I now move on to the discussion of verifying whether a given block  $T$  satisfies the requirements imposed by *coarse tenability*. Much of this discussion is based on the work of [Jansen and Vermeulen 2001] who described a method of verifying whether a set is *stable* under the definition of [Kohlberg and Mertens 1986].

Consider a block  $T$  with the property that all equilibria of the block game  $G^T$  are also equilibria of the entire bimatrix game  $G$ . I am interested in determining whether it is coarsely tenable. This entails determining how the equilibria with support contained in the block  $T$  change as the distribution over types  $\mu$  places small probability on blocks other than  $T$ . Lemma 1 implies that it suffices to consider deviations from convention  $T$  in which players consider individual pure strategies rather than larger strategy blocks. Lemmas 2 and 3 show that this problem can, with some care, be phrased as a problem about profiles in strategically perturbed games. Using the methodology of [Jansen and Vermeulen 2001], sub-blocks of  $T$  can be used to determine precisely how the equilibria of the perturbed game change, and thus how the equilibria of the consideration set game change as a result of altering the distribution over type. In what follows in Lemma 5 as well as throughout the rest of this paper,  $I$  refers to an indexed set of strategies in  $S_1$ . I sometimes use  $i \in I$  to refer to a strategy  $s_i \in I$  within this indexed set. I let  $p_i = \sigma_i(s_i)$  for a strategy  $s_i \in S_i$  and  $e_i$  denotes a

basis vector corresponding to playing strategy  $s_i$  with probability 1. Similar notation is used for an indexed set  $J \subseteq S_2$  and strategies for player 2.

**Lemma 5.** Consider a fixed strategy block  $T$ . A profile  $(p, q)$  is a  $T$ -equilibrium for the perturbed (bimatrix game)  $G = (N, S, u, \delta)$  if and only if there is a nonempty sub-block  $I \times J$  with  $I \subseteq T_1$  and  $J \subseteq T_2$  such that  $(p, \delta_1) \in S_{IJ}^1$  and  $(q, \delta_2) \in S_{IJ}^2$ , where

$$S_{IJ}^1 = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T_2 \\ p_i \geq \delta_1(s_i) & \forall i \in I \\ p_i = \delta_1(s_i) & \forall i \notin I \\ 0 \geq -\delta_1(s_i) & \forall i \\ \sum_i p_i = 1 \end{cases}$$

$$S_{IJ}^2 = \begin{cases} e_iAq - e_kAq \geq 0 & i \in I, k \in T_1 \\ q_j \geq \delta_2(s_j) & \forall j \in J \\ q_j = \delta_2(s_j) & \forall j \notin J \\ 0 \geq -\delta_2(s_j) & \forall j \\ \sum_j q_j = 1 \end{cases}$$

Since Lemmas 1-3 established a correspondence between  $T$ -equilibria and projections of equilibria of consideration-set games, every solution  $(p, q)$  to the above system given  $\delta$  yields an equilibrium of a consideration-set game in which  $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$  for each player  $i$ . For the case in which  $|T_i| \geq 2$  for each player, the solutions yield equilibria in which  $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ .

For a fixed sub-block  $I \times J$ ,  $S_{IJ}^1$  describes a convex polytope in  $\mathbb{R}^{2m}$ , and  $S_{IJ}^2$  describes a convex polytope in  $\mathbb{R}^{2n}$ . To determine whether a given strategy block  $T$  is coarsely tenable, it suffices to enumerate only those sub-block  $I \times J$  which possess a solution at  $\delta = 0$ . Indeed, the fact that equilibria of the incomplete information game induced by a type distribution  $\mu$  converge to equilibria of the block game  $G^T$  as  $\mu(T) \rightarrow 1$  implies that these are the only sub-blocks that are relevant to the problem. Any pair  $I \times J$  which does not possess a solution tuple at  $\delta = 0$  is such that there is no equilibrium of the block game in which each of the strategies in  $I$  and  $J$  are best responses. Then at least one of the associated polytopes is bounded away from the zero perturbation (full information) region. Since coarse tenability imposes restrictions only on perturbations sufficiently close to 0, such pairs  $I \times J$  can safely be ignored.

Given a concise description of the equilibria of the full game and that  $T$  is such that each equilibria of the block game  $G^T$  are also equilibria of the full game, it is not difficult to enumerate the pairs  $I \times J \subseteq T$  for which there are solutions at  $\delta = 0$ . Fix such a pair  $I \times J$ . In order to verify whether coarse tenability is violated, it suffices to determine whether there is a sequence of solutions  $\{(p_k, q_k, \delta_k)\}_{k=1}^\infty$  with  $\delta_k \rightarrow 0$  and such that for each  $(p_k, q_k)$ , there is some player  $i$  and strategy  $s_k \in S_i \setminus T_i$  such that  $s_k$  does strictly better than all strategies in  $T_i$ . The Bolzano-Weierstrass theorem implies that there exists a sequence in which the same strategy  $s_k = s^*$  plays this role. Such a strategy, if it exists, must then be a (weak) best reply to a solution of the system when  $\delta = 0$ . Let us suppose for the moment that such a strategy  $s^*$  exists and without loss of generality that it belongs to player 2, and denote its index in  $S_2$  by  $\omega$ . Fix some strategy  $s_j \in J$ . Evidently the above sequence must lie entirely within the half-space  $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$ .<sup>2</sup> Using the fact that the hyperplane  $\{(p, \delta) : pBe_\omega - pBe_j = 0\}$  intersects  $S_{IJ}^1$  at  $\delta = 0$  and the fact that a sequence of

<sup>2</sup>Note that implicitly this uses the fact that each player must be indifferent between all elements of  $I$  and  $J$  at any solution of the system.



solutions in which  $s^*$  does strictly better than  $s_j$  exists, it must be that  $S_{IJ}^1$  possesses a vertex that lies in the half space  $pBe_\omega - pBe_j > 0$ . In fact, this is the only instance in which such a sequence may exist. If all vertices lie in the half space  $pBe_\omega - pBe_j \leq 0$ , then the hyperplane  $pBe_\omega - pBe_j = 0$  is either tangent to  $S_{IJ}^1$  at  $\delta = 0$ , or aligns with one of its faces. In either case,  $s^*$  does not constitute a violation of coarse tenability.

**Proposition 3.** Fix a strategy block  $T$  with the property that all equilibria of the block game  $G^T$  are equilibria of the entire game  $G$ . Enumerate all sub-blocks  $I \times J$  of  $T$  for which  $S_{IJ}^1$  and  $S_{IJ}^2$  possess solutions when  $\delta = 0$ . For each such sub-block and for each player, enumerate all strategies outside of  $T$  that are (weak) best responses to an equilibrium with support contained within  $I \times J$ , denote the set of such strategies as  $\{\Omega_{IJ}^1, \Omega_{IJ}^2\}$ .

**Claim:**  $T$  is coarsely tenable if and only if for each such sub-block  $I \times J$ :

- (1) For each strategy  $s_\omega \in \Omega_{IJ}^2$ ,  $S_{IJ}^1$  does not possess a vertex that lies in the half-space  $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$ , where  $\omega$  denotes the index of  $s_\omega$  in  $S_2$  and  $j$  denotes the index of an arbitrary strategy in  $J$ .
- (2) For each strategy  $s_\omega \in \Omega_{IJ}^1$ ,  $S_{IJ}^2$  does not possess a vertex that lies in the half-space  $\{(q, \eta) : e_\omega Aq - e_i Aq > 0\}$ , where  $\omega$  denotes the index of  $s_\omega$  in  $S_1$  and  $i$  denotes the index of an arbitrary strategy in  $I$ .

This characterization of coarse tenability resembles the *undominated* characterization of perfect equilibria in bimatrix games due to van Damme (see Theorem 3.2.2 of [van Damme 1983]).

## 5 VERIFICATION OF FINELY TENABLE BLOCKS

Recall the definition of fine tenability introduced in Section 2 before continuing. In order to determine whether a given block  $T$  is finely tenable, it often suffices to show that it is coarsely tenable. Indeed, [Myerson and Weibull 2015] show that the concepts of coarse and fine tenability agree for generic normal-form games. While this is a useful result, much of the usefulness of fine tenability as a solution concept stems from its bite in normal form representations of extensive form games. Furthermore, determining whether or not a game is of a given generic form, for instance the class of *nondegenerate* games or the class of *hyper-regular* games introduced in [Myerson and Weibull 2015], can be computationally expensive. With this in mind, a discussion on fine tenability seems appropriate. Fortunately, Proposition 2 has already supplied a useful characterization for checking the conditions of fine tenability. In order to verify whether a given block  $T$  is finely tenable, it is helpful to return to the sets  $S_{IJ}^1$  and  $S_{IJ}^2$ . It once again suffices to consider only those sub-blocks  $I \times J$  of  $T$  that possess solutions when perturbations are set to zero.<sup>3</sup>

Fix such a sub-block  $I \times J$ . I am interested in determining whether there exists a sequence  $\epsilon_k \rightarrow 0$  and a sequence of  $T(\epsilon_k)$  – *proper* tuples  $(\sigma_k, \delta_k)$  which solve the systems  $S_{IJ}^1$  and  $S_{IJ}^2$  and are such that there is some player  $i$  and  $s_\omega \in S_i \setminus T_i$  with  $u_i(s_\omega, \sigma_{-i,k}) > \max_{t_i \in T_i} u_i(t_i, \sigma_{-i,k})$  for all  $k$ . Such a sequence  $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^\infty$  will be called an  *$I \times J$ -refutation*. If no such sub-block possesses a refutation, then  $T$  is finely tenable.

**Proposition 4.** A strategy block  $T$  is finely tenable if and only if every sub-block  $I \times J$  of  $T$  does not possess an  *$I \times J$ -refutation*.

Proposition 4 implies that the problem of verifying whether a block  $T$  is finely tenable reduces to the problem of finding  *$I \times J$ -refutations* among particular sub-blocks of  $T$ . However, this does not provide any indication of how to determine whether such refutations exist for a fixed  $T$ . Evidently,

<sup>3</sup>Note that while coarse tenability imposes the restriction that block equilibria are also equilibria of the entire game, fine tenability does not require this. See for instance Example 2 of [Myerson and Weibull 2015]

if an  $I \times J$  refutation exists, then it is possible to construct a refutation with the desirable property that the total preorders  $\succeq_{i,k}$  induced by  $\sigma_k$  are constant for all  $k$ . This suggests that a geometric approach to the problem of searching for refutations could be fruitful.

Fix a sub-block  $I \times J$  of the strategy block  $T$ . For a given total preorder  $\succeq_2$  for player 2, the set of strategies of player 1 that induce this preorder is given by  $p(\succeq_2)$ . Note that in general  $p(\succeq_2)$  is not closed, however the closure of this set is a polytope. Under the terminology of [Jansen 1993], the strategies in  $p(\succeq_2)$  are said to be *order equivalent*. Indeed, the space of mixed strategies for each player can be partitioned into equivalence classes using *order equivalence* as an equivalence relation.

Now, suppose there is an  $I \times J$ -refutation  $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^\infty$  with constant total preorders  $\succeq_1, \succeq_2$ . Then  $\sigma_{1,k} \in p(\succeq_2)$  and  $\sigma_{2,k} \in q(\succeq_1)$  for all  $k$ . Furthermore,  $\delta_{1,k}$  is  $\epsilon_k$ -proper according to  $\succeq_1$  for all  $k$ . Then it must be that  $S_{IJ}^1 \cap p(\succeq_2)$  and  $S_{IJ}^2 \cap q(\succeq_1)$  are nonempty and contain the perturbations  $\{\delta_{1,k}\}_{k=1}^\infty$  and  $\{\delta_{2,k}\}_{k=1}^\infty$  respectively.<sup>4</sup> The question is then: given a set  $S_{IJ}^1 \cap p(\succeq_2)$ , according to what total preorders  $\succeq_1$  does this set contain  $\epsilon$ -proper perturbations  $\delta_1$  for all  $\epsilon > 0$ ? Indeed, an answer to this question would provide a brute force method to the problem of verifying fine tenability. Namely, check all possible pairings of total preorders and determine whether the corresponding perturbations exist within  $S_{IJ}^1$  and  $S_{IJ}^2$  for all  $\epsilon > 0$ . The following results characterize necessary and sufficient conditions on  $S_{IJ}^1 \cap p(\succeq_2)$  to guarantee that it supports  $\epsilon$ -proper perturbations of a given total preorder  $\succeq_1$  for all  $\epsilon > 0$ . In what follows, the expression  $v(\delta_1(s))$  denotes the value of  $\delta_1(s)$  at a vertex  $v$  of a polytope.

**Lemma 6.** Fix a sub-block  $I \times J$  of a block  $T$ ,  $\succeq_1$  and  $\succeq_2$ . The set  $S_{IJ}^1 \cap p(\succeq_2)$  contains  $T(\epsilon)$ -proper tuples  $(\sigma_1, \delta_1)$  according to the order  $\succeq_1$  for all  $\epsilon > 0$  only if the following conditions hold

- There is a vertex of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  that lies on the zero perturbation region (i.e.  $v(\delta_1(s)) = 0$  for all  $s \in S_1$ ).
- For each strategy  $s \in S_1$ , there is a vertex of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  in which  $v(\delta_1(s')) = 0$  and  $v(\delta_1(s)) > 0$  for all  $s' \prec_1 s$ .

Before proving sufficiency, I will require the following lemma.

**Lemma 7.** For a fixed sub-block  $I \times J$  of a block  $T$  and total preorders  $\{\succeq_1, \succeq_2\}$ ,

$$relint(cl(S_{IJ}^1 \cap p(\succeq_2))) \subseteq S_{IJ}^1 \cap p(\succeq_2)$$

where  $relint(\cdot)$  denotes the relative interior of a set (under the usual topology).

**Lemma 8.** The conditions of Lemma 6 are sufficient to guarantee that for  $\epsilon > 0$ , there exists a tuple  $(\sigma_1, \delta_1)$  that is  $T(\epsilon)$ -proper according to  $\succeq_1$  and that lies in the relative interior of  $cl(S_{IJ}^1 \cap p(\succeq_2))$ .

With these results about the geometry of  $T(\epsilon)$ -proper tuples, I am able to provide an exact method for determining the presence of  $I \times J$ -refutations for any sub-block of  $T$ . Before describing the algorithm, I provide some basic definitions of the relevant objects used in the procedure.

**Definition 8.** The hyperplane arrangements, in  $\mathbb{R}^{2|S^1|}$  and  $\mathbb{R}^{2|S^2|}$  respectively, associated with the block  $I \times J$  are defined as

$$H_{IJ}^1 = \left\{ \sum_i p_i = 1 \right\} \cup \left\{ \bigcup_{\{i,j \in S^2, i < j\}} \{pBe_i - pBe_j = 0\} \right\} \cup \left\{ \bigcup_{i \in S^1} \{\delta_i = 0\} \cup \{p_i = \delta_i\} \right\}$$

<sup>4</sup>Here,  $p(\succeq_2)$  and  $q(\succeq_1)$  are taken to be those elements  $(p, \delta_1)$  and  $(q, \delta_2)$  in which  $p$  induces  $\succeq_2$  over  $S_2$  and  $q$  induces  $\succeq_1$  over  $S_1$ .

$$H_{IJ}^2 = \left\{ \sum_j p_j = 1 \right\} \cup \left\{ \bigcup_{\{i,j \in S^1, i < j\}} \{e_i Aq - e_j Aq = 0\} \right\} \cup \left\{ \bigcup_{j \in S^2} \{\eta_j = 0\} \cup \{q_j = \eta_j\} \right\}$$

The vertices of these hyperplane arrangements will be useful when computing vertices of the polytopes used in the procedure. Since it is unknown whether vertex enumeration can be done in polynomial time, it seems sensible to compute the vertices of this arrangement once and use them to quickly compute the vertices of one of the (possibly exponentially many) polytopes that are used in the procedure, rather than perform vertex enumeration at each step. Indeed, given a total preorder  $\succeq_2$ , the vertices of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  will be a subset of the vertices of  $H_{IJ}^1$ .

**Definition 9.** The zero perturbation regions of  $S_{IJ}^1$  and  $S_{IJ}^2$  projected onto the space of mixed strategies for each player are given by

$$Z_{IJ}^1 = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T^2 \\ p_i \geq 0 & \forall i \in I \\ p_i = 0 & \forall i \in S^1 \setminus I \\ \sum_i p_i = 1 \end{cases}$$

$$Z_{IJ}^2 = \begin{cases} e_i Aq - e_k Aq \geq 0 & i \in I, k \in T^1 \\ q_j \geq 0 & \forall j \in J \\ q_j = 0 & \forall j \in S^2 \setminus J \\ \sum_j q_j = 1 \end{cases}$$

These sets are used to prune the number of total preorders used in the procedure. In particular, it suffices to only focus on those total preorders which are locally possible given  $Z_{IJ}^1$  and  $Z_{IJ}^2$ . For instance, if  $Z_{IJ}^1$  is a singleton then any strict preferences in the total preorder  $\succeq_2$  induced at this point will be preserved as  $\epsilon \rightarrow 0$ . The algorithm is outlined on the next page.

**Remark 2.** It is important to note what is meant by "associated inequalities" in step 6(a) below. The associated inequalities are those found by relaxing all strict inequalities imposed by the total preorders into weak inequalities. In order to ensure that this would in fact result in the appropriate closures, it suffices to check that  $S_{IJ}^1 \cap p(\succeq_2)$  and  $S_{IJ}^2 \cap q(\succeq_1)$  are nonempty. This can of course be done by checking any convex combination of the vertices found in step 6(a) in which all vertices receive strictly positive weight.

**Proposition 5.** Algorithm 1 correctly determines in finite time whether the sub-block  $I \times J$  of  $T$  possesses a refutation.

A number of comments on the method are in order. It is in principle possible to verify whether a pair of total preorders  $\{\succeq_1, \succeq_2\}$  constitute a refutation by solving a pair of linear programs (see lemma 13). On the other hand, enumerating all total preorder pairs and performing this procedure very quickly becomes infeasible. The first 5 steps of the procedure will in general (at least in nondegenerate games) drastically reduce the number of pairs to check. It is of course possible to refine and improve upon this pruning of the total preorders by using the structure of the face lattice of the associated hyperplane arrangements alongside the restrictions imposed by properness. These improvements deserve some attention, and are a subject of ongoing research.

Algorithm 1 here is in some sense a formal generalization of the informal method of analytically determining whether a given equilibrium is proper given in [Myerson 1991] and fleshed out a bit more in [Belhaiza et al. 2012]. The method here is adapted to a different solution concept, but the logic behind lemmas 6-8 and the procedure defined above are applicable to the problem of

verifying proper equilibria. The formalization given here does provide a number of insights into the geometry of the problem, however.

The algorithm described here is exponential in the worst case, requiring the comparison of an exponential number of total preorder pairings. The next result shows that it is NP-hard to determine whether a block possesses a refutation, even when it consists of single strategies. The following reduction modifies the one provided by [Hansen and Lund 2018], who in turn modify a reduction given by [Conitzer and Sandholm 2008]. I construct a game in which a given block possesses a refutation if and only if an instance of a Boolean satisfiability problem in conjunctive normal form with exactly three literals per clause possesses no satisfying assignment.

### Algorithm 1: Search for IJ-refutations

- (1) Compute the vertices  $V_H^i$  of  $H_{IJ}^i$  and  $V_Z^i$  of  $Z_{IJ}^i$  using a vertex enumeration algorithm for hyperplane arrangements, e.g. [Avis and Fukuda 1992]
- (2) Set  $j_1 \sim_2 j_2 \quad \forall j_1, j_2 \in J$  and  $i_1 \sim_1 i_2 \quad \forall i_1, i_2 \in I$
- (3) For each  $k_1, k_2 \in S_2 \setminus J, k_1 \neq k_2$ :
  - (a) If  $V_Z^1 \subset pBe_{k_1} - pBe_{k_2} > 0$ , set  $k_1 >_2 k_2$
  - (b) Else if  $V_Z^1 \subset pBe_{k_1} - pBe_{k_2} < 0$ , set  $k_2 >_2 k_1$
  - (c) Else place no restriction on  $k_1$  and  $k_2$  regarding  $\succeq_2$
- (4) Print restrictions on  $\succeq_2$
- (5) Repeat steps 3 – 4 for restrictions on  $\succeq_1$
- (6) For each pair of total preorders  $\succeq_1$  and  $\succeq_2$  satisfying the restrictions from steps 2 – 5 and such that there is some player  $i$  and strategy  $s_\omega \in S_1(S_2)$  with  $s_\omega >_i s$  for some  $s \in I(J)$ 
  - (a) Enumerate the vertices of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  and  $cl(S_{IJ}^2 \cap q(\succeq_1))$  by checking which vertices of  $H_{IJ}^1$  and  $H_{IJ}^2$  satisfy the associated inequalities.
  - (b) Check whether the vertices of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  satisfy the conditions of Lemma 6 with respect to  $\succeq_1$  to determine existence of  $T(\epsilon)$  – *proper* perturbations for  $\epsilon$  near zero. Repeat for  $cl(S_{IJ}^2 \cap q(\succeq_1))$  with respect to  $\succeq_2$ .
  - (c) If the requisite conditions are satisfied for both  $cl(S_{IJ}^1 \cap p(\succeq_2))$  and  $cl(S_{IJ}^2 \cap q(\succeq_1))$  then a refutation has been found, and the procedure halts after printing the pairing of total preorders  $\{\succeq_1, \succeq_2\}$ .
  - (d) Else there is no refutation for the total preorder pairing
- (7) If no pairing of total preorders yields a refutation, then the block  $I \times J$  possesses no refutations.

**Definition 10.** Let  $\phi$  be a Boolean formula in conjunctive normal form with exactly three literals per clause. Let  $V$  denote the set of variables, with  $|V| = n$ . Let  $L$  denote the set of literals, with  $+l_i$  and  $-l_i$  denoting the positive and negative literals corresponding to variable  $x_i$ . Let the mapping  $v : L \rightarrow V$  be defined by  $v(+l_i) = v(-l_i) = x_i$ , assigning to any literal its corresponding variable. Denote the set of clauses by  $C$ . Finally, each player will have three additional pure strategies denoted by  $f, g$ , and  $h$ . The symmetric, normal form game  $G(\phi)$  that I consider is defined, with payoffs to the row player, in Table 1 below. The block to be considered is  $T = \{g\} \times \{g\}$ . In what follows, I will show that  $T$  is finely tenable if and only if the formula  $\phi$  does not have a satisfying assignment.

	$l'$	$x'$	$h'$	$c'$	$f'$	$g'$
$l$	$n - 1$ if $l \neq -l'$ $n - 4$ if $l = -l'$	$n - 1$ if $v(l) \neq v$ $n - 4$ if $v(l) = v$	$n - 1$	$n - 1$	$n - 1$	$n - 3$
$x$	$n - 1$ if $v(l) \neq v$ $n$ if $v(l) = v$	$n - 1$	$n - 1$	$n - 1$	$n - 1$	$n - 2$
$h$	$n - 1 - \frac{1}{2n}$	$\frac{(n-1)^2+(n-4)}{n}$	$n - 1$	$n - 1$	$n - 1$	$n - 3$
$c$	$1$ if $l \notin c$ $0$ if $l \in c$	$n - 1$	$n$	$n - 1$	$n - 1$	$n - 1$
$f$	$\frac{n-\frac{1}{2}}{n}$	$n - 1$	$n - 1$	$n - 1$	$n - 1$	$n - 1$
$g$	$n - 1$	$n - 1$	$n - 1$	$n$	$0$	$n - 1$

Table 1. The game  $G(\phi)$ 

**Lemma 9.** Let  $(\sigma_1, \sigma_2)$  denote any  $T(\epsilon)$  – *proper* profile for  $\epsilon$  small. Then for each player  $i$  and each pair of variables  $x_j$  and  $x_k$ ,

$$\sigma_i(+l_j) + \sigma_i(-l_j) = \sigma_i(+l_k) + \sigma_i(-l_k)$$

and

$$\sigma_i(x_j) = \sigma_i(x_k)$$

**Lemma 10.** If  $(\sigma_1, \sigma_2)$  is a  $T(\epsilon)$  – *proper* profile with  $\epsilon$  small, then there is a partition of the set of literals  $L$  into two sets  $L_1$  and  $L_2$  such that for each player

$$\sigma_i(l_1) = \sigma_i(l'_1) \quad \forall l_1, l'_1 \in L_1$$

$$\sigma_i(l_2) = \sigma_i(l'_2) \quad \forall l_2, l'_2 \in L_2$$

$$l_j \in L_1 \Rightarrow -l_j \in L_2$$

$$l_1 \succeq_i l_2 \quad \forall l_1 \in L_1, l_2 \in L_2$$

**Lemma 11.** If  $\phi$  has a satisfying assignment then  $T = \{g\} \times \{g\}$  possesses a refutation.

**Lemma 12.** If  $T = \{g\} \times \{g\}$  possesses a refutation, then  $\phi$  has a satisfying assignment.

**Proposition 6.** It is NP-hard to verify whether a given sub-block  $I \times J$  of a block  $T$  possesses an  $I \times J$ -*refutation*.

Note that proposition 4 alongside lemmas 6-8 show that one can in fact view  $I \times J$ -*refutations* as a pair of total preorders over the pure strategies of each player. If one is provided with a pair of total preorders, one for each player, it is not difficult to determine whether it serves as an  $I \times J$  refutation. This provides a proof for NP-completeness of finding refutations.

**Lemma 13.** Given a block  $T$ , a sub-block  $I \times J$ , and a pair of total preorders  $\succeq_1, \succeq_2$ , one can determine whether  $\{\succeq_1, \succeq_2\}$  serves as an  $I \times J$  refutation for  $T$  by solving a polynomial number of linear programs, each of which contains only a polynomial number of constraints.

**Proposition 7.** The problem of verifying whether a given sub-block  $I \times J$  of  $T$  possesses an  $I \times J$ -*refutation* is NP-complete.

## 6 APPLICATION TO PROPER EQUILIBRIA

The results of Lemmas 6-8, used here to analyze fine tenability, can also be applied to proper equilibria. As I will soon show, it gives rise to an exact method of verifying whether a given equilibrium constitutes a proper equilibrium of a bimatrix game. Before proceeding, I introduce some basic definitions for the section.

**Definition 11.** [Myerson 1978] A mixed strategy profile  $\sigma$  is said to be an  $\epsilon$ -proper equilibrium if for each player  $i$  the following conditions hold:

- $\sigma_i(s) > 0 \quad \forall s \in S_i$
- $u_i(s, \sigma_{-i}) > u_i(s', \sigma_{-i}) \Rightarrow \sigma_i(s)\epsilon \geq \sigma_i(s') \quad \forall s, s' \in S_i$

A Nash equilibrium  $\sigma$  is said to be a *proper equilibrium* if there exists a sequence  $\{\epsilon_k, \sigma^k\}_{k=1}^\infty$  such that

- $\epsilon_k > 0$  and  $\lim_{k \rightarrow \infty} \epsilon_k = 0$
- $\sigma^k$  is an  $\epsilon_k$ -proper equilibrium
- $\lim_{k \rightarrow \infty} \sigma^k = \sigma$

The procedure below determines whether a given input Nash equilibrium  $\sigma = (p^*, q^*)$  is a proper equilibrium. It is based on the logic of Lemmas 6-8 above that it is possible to determine whether a convergent sequence of  $\epsilon$ -proper equilibria exists by analyzing the faces of a pair of hyperplane arrangements. Define the hyperplane arrangements:

$$\begin{aligned} \tilde{H}^1 &= \left\{ \sum_i p_i = 1 \right\} \cup \left\{ \bigcup_{\{j,k \in S^2: j < k\}} \{pBe_j - pBe_k = 0\} \right\} \cup \left\{ \bigcup_i \{p_i = 0\} \right\} \\ \tilde{H}^2 &= \left\{ \sum_j q_j = 1 \right\} \cup \left\{ \bigcup_{\{i,k \in S^1: i < k\}} \{e_iAq - e_kAq = 0\} \right\} \cup \left\{ \bigcup_j \{q_j = 0\} \right\} \end{aligned}$$

Consider the following sets:

$$\begin{aligned} P^1(p^*, q^*) &= \begin{cases} pBe_j - pBe_k \geq 0 & j \in Y(q^*), k \in S_2 \\ p_i \geq 0 & \forall i \in S^1 \\ \sum_i p_i = 1 \end{cases} \\ P^2(p^*, q^*) &= \begin{cases} e_iAq - e_kAq \geq 0 & i \in Y(p^*), k \in S_1 \\ q_j \geq 0 & \forall j \in S^2 \\ \sum_j q_j = 1 \end{cases} \end{aligned}$$

where  $Y(\cdot)$  denotes the carrier of a mixed strategy. Each closed face (of positive dimension)  $f_1$  of  $\tilde{H}^1$  such that  $p^* \in f_1$  and  $f_1 \subseteq P^1(p^*, q^*)$  is associated with a total preorder  $\succeq_2$ , the preference ordering of player 2 over strategies in  $S^2$  induced by mixed strategies that lie in the relative interior of  $f_1$ . A similar statement holds for faces  $f_2$  of  $\tilde{H}^2$ . To determine whether  $(p^*, q^*)$  is a proper equilibrium, it suffices to check all pairings of such faces  $f_1$  and  $f_2$  for a set of conditions similar to those given in Lemma 6. In what follows, let  $\succeq_{i,f_i}$  denote the total preorder over strategies for player  $i$  induced by strategies on the relative interior of  $f_i$ . For a given total preorder  $\succeq_{i,f_i}$ , let  $\zeta_k^i$  denote the  $k^{\text{th}}$  indifference class, with  $\zeta_1^i$  denoting the best strategies for player  $i$ . Let  $r_i$  denote the total number of indifference classes in  $S_i$  according to the total preorder  $\succeq_i$ .

**Lemma 14.** A Nash equilibrium  $\sigma = (p^*, q^*)$  is a *proper equilibrium* if and only if there is a pair of faces  $f_1$  (of  $\tilde{H}^1$ ) and  $f_2$  (of  $\tilde{H}^2$ ) such that

- $p^* \in f_1$  and  $q^* \in f_2$
- $f_1 \subseteq P^1(p^*, q^*)$  and  $f_2 \subseteq P^2(p^*, q^*)$
- For each player  $i$ , each indifference class  $\zeta_k^i$  and each strategy  $s \in \zeta_k^i$ , there is a vertex of  $f_i$  in which  $p_s > 0$  and  $p_{s'} = 0$  for all  $s' \in \zeta_l^i$  with  $l > k$ .

## 7 DISCUSSION

The characterization in Section 3 provides a straightforward method of verifying coarse tenability of a given block  $T$  regardless of degeneracy assumptions on the game. A natural method of finding coarsely tenable blocks is then to determine all strategy blocks of the game which possess the property that all block equilibria are also equilibria of the full game, and then determining whether the block satisfies the additional conditions imposed by coarse tenability. Enumerating such blocks for nondegenerate games is straightforward, albeit computationally expensive. The relationship between projections of equilibria of consideration-set games and the usual notion of strategic perturbations has allowed me to relate coarse tenability to KM-stable sets. It would be interesting to explore the relationship between tenable strategy sets and other familiar solution concepts in the game theory literature (see Wikman (2016) for work in this area).

The methods developed in Section 4 are based on simple geometric arguments about *proper* strategy profiles. Note that the procedures for verifying the conditions of fine tenability and proper equilibria, while exact, may require calculations involving extremely small numbers, and bounds for these values can be articulated using the geometry of bimatrix games; although, it may be computationally expensive to do so. This is the main drawback of the methods proposed here. On the other hand, it is quite useful to have bounds on the level of precision required for verifying these concepts. For instance, the results of this paper can be applied to the mixed 0-1 quadratic programming problem proposed in [Belhaiza et al. 2012] for the verification of proper equilibria.

There are a number of avenues in which the current paper can be improved. First, it would be very interesting to have a procedure which computes minimal strategy sets that satisfy the criteria of the two solution concepts discussed in this paper, even for bimatrix games. In particular, a procedure that is motivated by a population of agents collectively steering toward a stable convention seems the most promising. Note that, unlike sets which are closed under rational belief (CURB), the intersection of two coarsely (finely) tenable blocks need not be coarsely (finely) tenable in bimatrix games. This appears to preclude a procedure along the lines of [Benisch et al. 2010] in which a minimal block can be constructed by iteratively including additional strategies, as now the final product may not be minimal under set-inclusion (this is due to the fact that the ordering of the addition of new strategies to the block is now important).

It is worth noting that complexity results for coarse tenability are also available, and NP-hardness of the problem of verifying coarse tenability can be proved by a straightforward modification of the reduction given in [Conitzer and Sandholm 2008]. A formal procedure for verifying coarse tenability is also a natural next step in the analysis. The procedure given here for fine tenability can also be improved upon a great deal. As mentioned in section 5, it is possible to use the properties of the face semi-lattice of the associated hyperplane arrangement to reduce the number of total preorders enumerated. The exact nature of this improvement is a subject of ongoing research.

## REFERENCES

- David Avis and Komei Fukuda. 1992. A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra. *Discrete and Computational Geometry* 8, 3 (Sept. 1992), 295–313.
- Davis Avis, Gabriel Rosenberg, Rahul Savani, and Bernhard von Stengel. 2010. Enumeration of Nash Equilibria for Two-Player Games. *Economic Theory* 42, 1 (Jan. 2010), 9–37.
- Kaushik Basu and Jörgen Weibull. 1991. Strategy Subsets Closed Under Rational Behavior. *Economics Letters* 36, 2 (June 1991), 141–146.
- Slim Belhaiza, Charles Audet, and Pierre Hansen. 2012. On Proper Refinement of Nash Equilibria for Bimatrix Games. *Automatica* 48, 2 (Feb. 2012), 297–303.
- Michael Benisch, George David, and Tuomas Sandholm. 2010. Algorithms for Closed Under Rational Behavior (CURB) Sets. *Journal of Artificial Intelligence Research* 38 (Aug. 2010), 513–534.
- Dimitri Bertsekas, Angelia Nedic, and Asuman Ozdaglar. 2003. *Convex Analysis and Optimization*. Athena Scientific, Belmont, MA.
- Felix Brandt and Markus Brill. 2016. Computing Dominance-Based Solution Concepts. *ACM Transactions on Economics and Computation* 5, 2 (Oct. 2016).
- Xi Chen and Xiaotie Deng. 2006. Settling the Complexity of Two-Player Nash Equilibrium. In *Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS '06)*. ACM, Berkeley, CA.
- Vincent Conitzer and Tuomas Sandholm. 2008. New Complexity Results About Nash Equilibria. *Games and Economic Behavior* 63, 2 (July 2008), 621–641.
- Constantinos Daskalakis, Paul Goldberg, and Christos Papadimitrou. 2006. The Complexity of Computing a Nash Equilibrium. In *Proceedings of the Annual Symposium on Theory of Computing (STOC '06)*. ACM, Seattle, WA.
- John Duggan and Michel Le Breton. 2014. *Choice-Theoretic Solutions for Strategic Form Games*. Retrieved February 10, 2020 from <https://sites.google.com/view/johnduggan/other-papers?authuser=0>
- Itzhak Gilboa and Eitan Zemel. 1989. Nash and Correlated Equilibria: Some Complexity Considerations. *Games and Economic Behavior* 1, 1 (March 1989), 80–93.
- Kristoffer Hansen. 2017. The Real Computational Complexity of Minmax Value and Equilibrium Refinements in Multi-player games. In *Algorithmic Game Theory (LNCS) (SAGT '17)*. Springer, Cham.
- Kristoffer Hansen and Troels Lund. 2018. Computational Complexity of Proper Equilibrium. In *Proceedings of the 2018 ACM Conference on Economics and Computation (EC '18)*. ACM, New York, NY.
- Kristoffer Hansen, Peter Miltersen, and Troels Sørensen. 2010. The Computational Complexity of Trembling Hand Perfection and Other Equilibrium Refinements. In *Proceedings of the third international conference on Algorithmic Game Theory (SAGT '10)*. ACM, Athens, Greece.
- Mathijs Jansen. 1981. Maximal Nash Subsets for Bimatrix Games. *Naval Research Logistics Quarterly* 28, 1 (March 1981).
- Mathijs Jansen. 1993. On the Set of Proper Equilibria of a Bimatrix Game. *International Journal of Game Theory* 22, 2 (June 1993), 97–106.
- Mathijs Jansen and Dries Vermeulen. 2001. On the Computation of Stable Sets and Strictly Perfect Equilibria. *Economic Theory* 17, 2 (March 2001), 325–344.
- Max Klimm, Tuomas Sandholm, and Jörgen Weibull. 2011. *Computational Aspects of Closure Under Rational Behavior in Games*. <https://sites.google.com/site/joergenweibull/working-papers>
- Elon Kohlberg and Jean-Francois Mertens. 1986. On the Strategic Stability of Equilibria. *Econometrica* 54, 5 (Sept. 1986), 1003–1037.
- Peter Bro Miltersen and Troels Bjerre Sørensen. 2010. Computing a quasi-perfect equilibrium of a two-player game. *Economic Theory* 42, 1 (2010), 175–192.
- Roger Myerson. 1978. Refinements of the Nash Equilibrium Concept. *International Journal of Game Theory* 7, 2 (June 1978), 73–80.
- Roger Myerson. 1991. *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, MA.
- Roger Myerson and Jörgen Weibull. 2015. Tenable Strategy Blocks and Settled Equilibria. *Econometrica* 83, 3 (May 2015), 943–976.
- John Nash. 1951. Non-Cooperative Games. *Annals of Mathematics* 54, 2 (Sept. 1951), 286–295.
- Troels Sørensen. 2012. Computing a Proper Equilibrium of a Bimatrix Game. In *In ACM Conference on Electronic Commerce (EC '12)*. ACM, Valencia Spain.
- Eric van Damme. 1983. *Refinements of the Nash Equilibrium Concept*. Springer, New York, New York.
- John von Neumann and Oskar Morgenstern. 1944. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey.
- Peter Wikman. 2016. *Nash Blocks*. <https://sites.google.com/wikman.com/peter/research?authuser=0>



## A APPENDIX OF PROOFS

### Proof of Lemma 1

PROOF. Suppose that  $|T_i| \geq 2$  for each player  $i$ . Construct  $\mu'$  using the projection  $\tau^\mu$  as follows :

$$\begin{aligned}\mu'_i(T_i) &= \mu_i(T_i) \quad \forall i \\ \mu'_i(s_i) &= \sum_{\{C_i: s_i \in C_i, C_i \neq T_i\}} \tau_{i|C_i}(s_i) \mu_i(C_i) \quad \forall s_i \in S_i \text{ and } \forall i \\ |C_i| \neq 1 \wedge C_i \neq T_i &\Rightarrow \mu'_i(C_i) = 0\end{aligned}$$

Now  $\tau'$  has essentially been constructed for us. Given the restrictions imposed by  $\mu'$ ,  $\tau'_i$  is pinned down for all sets other than  $T_i$  for each player  $i$ . Finally, set  $\tau'_{i|T_i} = \tau_{i|T_i}$ . To see that the proof is complete, note that by construction  $\tau^\mu = \tau'^\mu$ , the equilibria induce the same distribution over pure strategies for each player. Therefore since  $\tau_{i|T_i}$  was part of an equilibrium, it must also be that  $\tau'_{i|T_i}$  is as well.

For the case in which  $T_i = s_i$  for some player  $i$ , define  $\mu'_i(T_i) = \tau^\mu(s_i)$ .  $\square$

### Proof of Lemma 2

PROOF. Fix a *simple* type distribution  $\mu$  and suppose for now that  $|T_i| \geq 2$  for each player  $i$ . Let  $\tau$  be an equilibrium of the consideration-set game  $G^\mu$  with associated projection  $\tau^\mu$ . Define  $\delta_i(s_i) = \mu_i(s_i)$  for each strategy  $s_i \in S_i$  and each player  $i$ . Define  $\sigma_i(s_i) = \tau^\mu(s_i)$ . To see that  $\sigma$  is a *T-equilibrium*, note that if  $s_i \notin BR_i^T(\sigma)$  then the fact that  $\tau$  is an equilibrium implies that  $\tau_{i|T_i}(s_i) = 0$  and hence  $\sigma_i(s_i) = \mu_i(s_i) = \delta_i(s_i)$  which implies that  $s_i \notin Y_i^\delta(\sigma_i)$ .

For the case in which  $|T_i| = 1$  for some player  $i$ , define  $\delta_i(s_i) = 0$  for such players.  $\square$

### Proof of Lemma 3

PROOF. Suppose that  $|T_i| \geq 2$  for each player  $i$ . Given  $\delta$ , set  $\mu_i(s_i) = \delta_i(s_i)$  for each  $s_i \in S_i$  and each player  $i$ . Set  $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$  for each player  $i$ . Define

$$\tau_{i|T_i}(s_i) = \frac{\sigma_i(s_i) - \delta_i(s_i)}{\mu_i(T_i)} \quad \forall s_i \in T_i$$

for each player  $i$ . The fact that  $\mu$  is assumed to be a *simple* type distribution implies that  $\tau$  is pinned down for all blocks aside from  $T$ . Thus by construction

$$\tau_i^\mu(s_i) = \sigma_i(s_i)$$

and furthermore the fact that  $\sigma$  is a *T-equilibrium* implies that all strategies  $s_i \in T_i$  such that  $\tau_{i|T_i}(s_i) > 0$  are an element of  $BR_i^T(\tau^\mu)$ .

For the case in which  $|T_i| = 1$  for some player  $i$ , set  $\mu_i(T_i) = 1 - \sum_{s_i \notin T_i} \delta_i(s_i)$ . Note that  $\tau_i$  is automatically constructed for these players since  $\mu$  is *simple*.  $\square$

### Proof of Proposition 1

PROOF. I will show that  $T$  contains a KM-set, and hence must contain a minimal KM-set. Let  $E$  denote the set of equilibria of the block game  $G^T$  (which are also equilibria of the entire game under the assumption of coarse tenability). Since  $E$  is the entire set of equilibria of a block game, it is closed.

Suppose, seeking contradiction, that this set  $E$  is not a KM-set. Then there is a neighborhood  $V$  of  $E$  in the space of mixed strategies such that for any  $\eta > 0$ , there is a perturbation  $\delta$  with  $\|\delta\| < \eta$  and such that all equilibria of the game perturbed under  $\delta$  lie outside of  $V$ . Thus there exists a sequence of perturbations  $\{\delta_k\}_{k=1}^\infty$  such that  $\|\delta_k\| > 0$  for each  $k$  and  $\|\delta_k\| \rightarrow 0$ , and such that for each  $k$ , the perturbed game  $(N, S, u, \delta)$  possesses no equilibria within  $V$ .

Suppose that  $|T_i| \geq 2$  for each player  $i$ . Interpret the perturbations  $\{\delta_{i,k}\}_{i=1}^N$  as a *simple* type distribution  $\mu_k$  with respect to  $T$  for a moment, with  $\mu_{i,k}(T_i) = 1 - \sum_{s_i \in S_i} \delta_{i,k}(s_i)$  and  $\mu_{i,k}(s_i) = \delta_{i,k}(s_i)$ . Recall that the projections of equilibria of the associated consideration-set games converge to equilibria of the block game  $G^T$  as  $\mu(T) \rightarrow 1$  (i.e.  $k \rightarrow \infty$ ). Hence there is a  $K_1 > 0$  such that for all  $k \geq K_1$ , there is an equilibrium of the consideration-set game induced by the *simple* distribution  $\mu_k$  that lies in the neighborhood of  $V$ . Now, if I can show that the projections of these equilibria of the consideration-set game induced by  $\mu_k$  are (eventually) also equilibria of the perturbed game  $(N, S, u, \delta)$ , the proof will be complete. This is precisely where the assumption of coarse tenability has bite. Indeed, there is  $\bar{\epsilon} > 0$  such that if  $\mu(T) > 1 - \bar{\epsilon}$ , then all projections of equilibria of the consideration-set game are also equilibria of the associated perturbed game. To see why, recall that the profile  $\sigma$  is an equilibrium of the perturbed game  $(N, S, u, \delta)$  if and only if  $\sigma$  satisfies the minimum probability constraints imposed by  $\delta$  and for each player  $i$ :

$$Y_i^\delta(\sigma_i) \subseteq BR_i(\sigma)$$

Let  $K_2 > 0$  be such that for all  $k \geq K_2$ ,  $\mu_k(T) > 1 - \bar{\epsilon}$ , and let  $K^* = \max\{K_1, K_2\}$ . Consider any equilibrium  $\tau$  of a consideration-set game  $\mu_k$  for  $k \geq K^*$ . Then

$$\tau_{i|T_i}(s_i) > 0 \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

since  $T$  is coarsely tenable. This then implies that

$$\tau_{i|T_i}^{\mu_k}(s_i) > \mu_k(s_i) = \delta_{i,k}(s_i) \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

Thus  $\tau^{\mu_k} = \sigma$  constitutes an equilibrium of the perturbed game  $(N, S, u, \delta)$  that lies within the set  $V$ , a contradiction.

For the case in which  $|T_i| = 1$  for some player  $i$ , define  $\mu_{i,k}(T_i) = 1 - \sum_{s_i \notin T_i} \delta_{i,k}(s_i)$  for such players. The rest of the argument is the same.  $\square$

#### Proof of Lemma 4

PROOF. Let  $|T_i| \geq 2$  for each player. For a player  $i$  and given strategy  $s_i \in S_i$ , let  $\beta_i(s_i) \subseteq \mathbb{P}(S_i)$  denote the collection of sets  $C_i \in \mathbb{P}(S_i)$  such that

$$s_i \in \operatorname{argmax}_{t_i \in C_i} u_i(t_i, \tau_{-i}^\mu)$$

It is important to distinguish between two cases:

- Case 1: There exist two strategies  $r_i, t_i \in S_i$  such that  $u_i(r_i, \tau_{-i}^\mu) < u_i(t_i, \tau_{-i}^\mu)$  and such that there is  $C_i^* \in \beta_i(r_i)$  with  $T_i = C_i^* \cup \{t_i\}$ .

In this case, note that  $t_i$  is the unique best response in  $T_i$ . Define

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i) \mu_i(C_i)$$

for all  $s_i \neq t_i$ . Define

$$\delta_i(t_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(t_i) \mu_i(C_i) + \frac{1}{\epsilon} \mu_i(C_i^*)$$

Define  $\sigma_i = \tau_{-i}^\mu$ . Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i)\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq C_i^*\}} \mu_i(C_i \cup \{t_i\}) + \mu_i(C_i^*) \leq \epsilon \delta_i(t_i)$$

Now let  $s_i$  be such that  $u_i(s_i, \tau_{-i}^\mu) > u_i(t_i, \tau_{-i}^\mu)$ . Then

$$\delta_i(t_i) \leq \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i) + \frac{1}{\epsilon} \mu_i(C_i^*) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i \cup \{s_i\}) + \epsilon \mu_i(T_i \cup s_i) \leq \epsilon \delta_i(s_i)$$

Verifying all other pairings uses the same logic, and is more simple due to the fact that  $T_i$  does not appear in the calculations. See case 2 for how this is done explicitly. Note that

$$\sum_{s_i \in S_i} \delta_i(s_i) = \sum_{C_i \neq T_i} \mu_i(C_i) + \frac{1}{\epsilon} \mu_i(C_i^*) \leq \epsilon^3 + \epsilon^2 \mu_i(T_i) \leq \epsilon$$

- Case 2: There do not exist two strategies  $r_i, t_i \in S_i$  such that  $u_i(r_i, \tau_{-i}^\mu) < u_i(t_i, \tau_{-i}^\mu)$  and such that there is  $C_i^* \in \beta_i(r_i)$  with  $T_i = C_i^* \cup \{t_i\}$ .

In this case, it suffices to set

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i) \mu_i(C_i)$$

Then let  $r_i, t_i \in S_i$  such that  $u_i(r_i, \tau_{-i}^\mu) < u_i(t_i, \tau_{-i}^\mu)$ . Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i \cup \{t_i\}) \leq \epsilon \delta_i(t_i)$$

The case in which  $|T_i| = 1$  for some player  $i$  only requires slight modifications, and only in the case in which the strategy  $s_i = T_i$  is among the *worst* strategies in  $S_i$ . In such a case, choose  $\delta_i(s_i)$  to be a sufficiently small positive number, and construct  $\delta_i$  the same way as above for all other strategies.

□

## Proof of Proposition 2

PROOF. Suppose without loss of generality that  $m = |S_1| = \max_i \{|S_i|\}$ . Fix  $\epsilon^* = \frac{1}{m^2 2^{2m}}$  and  $z = 2m$ . Let  $(\sigma, \delta)$  be a  $T(\epsilon)$ -proper tuple for  $\epsilon < \epsilon^*$ . I will construct a type distribution  $\mu$  that is  $\epsilon^{\frac{1}{z}}$ -proper as well as an equilibrium  $\tau$  of the associated consideration-set game directly. Enumerate the indifference classes  $\zeta_1^i, \zeta_2^i, \dots, \zeta_{r_i}^i$  induced by  $\sigma$  for each player  $i$ . For a given strategy  $s$ , let  $\zeta(s)$  denote the indifference class that strategy  $s$  is an element of. The construction below is for player 1, without any loss of generality.

I will distinguish between two cases.

- Case 1:  $T_1 \neq \bigcup_{l \leq k} \zeta_l^1$  for any  $k$ .

Define the term:

$$R(s) = \sum_{\{C \subset C \cup \{s' \preceq s\}, C \neq T_1 : s \in C\}} \frac{1}{\epsilon^{\frac{|C|-1}{z}} |C \cap \zeta(s)|}$$

Note that for strategies in the same indifference class,  $R$  can differ only if one is an element of  $T$  and the other is not. If  $|\zeta_1^1| = 1$ , let  $s_1$  denote its element and define  $\mu_1(s) = \delta_1(s) := \gamma_1^1$  for all  $s \in S^1$ . Otherwise let

$$\mu_1(s) = \min \left[ \frac{\min_{\{s \in \zeta_1^1\}} \delta_1(s)}{\max_{\{s \in \zeta_1^1\}} R(s)}, \frac{\sum_{s \in \zeta_1^1} \delta_1(s)}{\sum_{s \in \zeta_1^1} R(s) + \frac{1}{\epsilon^{\frac{|\zeta_1^1|-1}{z}}}} \right] := \gamma_1^1$$

for all  $s \in S^1$ . Set  $\mu_1(C) = \frac{\mu_1(s)}{\epsilon^{\frac{|C|-1}{z}}}$  for all  $C$  such that  $|C| < |\zeta_1^1|$  and  $C \neq T_1$ . Define

$$\mu_1(C) = \sum_{s \in \zeta_1^1} \delta_1(s) - \gamma_1^1 \sum_{s \in \zeta_1^1} R(s) := \gamma_2^1$$

for all  $C \neq T_1$  such that  $|C| = |\zeta_1^1|$ . For all subsets  $C$  such that  $|\zeta_1^1| < |C| < |\zeta_1^1| + |\zeta_2^1|$ , set  $\mu_1(C) = \frac{\gamma_2^1}{\epsilon^{\frac{|C|-|\zeta_1^1|}{z}}}$ . For any strategy  $s$ , define

$$\bar{R}(s) = \sum_{\{C \subset \cup_{\{s' \preceq s\}} s', C \neq T_1 : s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta(s)|}$$

Here,  $\bar{R}(s)$  can be interpreted as the probability that a strategy  $s$  receives from all blocks in which it is among the best alternatives, aside from the largest such block. This is of course under the assumption that the conditional distribution over the best alternatives in a block is uniform. When moving to the largest block in which a given strategy is a best alternative, the conditional distribution will no longer be uniform. This will be shown in the construction of  $\tau_1$  later on.

Let

$$\mu_1(C) = \sum_{s \in \zeta_2^1} \delta_1(s) - \sum_{s \in \zeta_2^1} \bar{R}(s) := \gamma_3^1$$

for all sets  $C \neq T_1$  such that  $|C| = |\zeta_1^1| + |\zeta_2^1|$ . Now it is possible to give an iterative construction of  $\mu_1$  for all subsets. For all sets  $C \neq T_1$  such that  $\sum_{l=1}^k |\zeta_l^1| < |C| < \sum_{l=1}^{k+1} |\zeta_l^1|$  for some  $k < r_1$ , define

$$\mu_1(C) = \frac{\gamma_{k+1}^1}{\epsilon^{\frac{|C| - \sum_{l=1}^k |\zeta_l^1|}{z}}}$$

where

$$\gamma_k^1 = \sum_{s \in \zeta_{k-1}^1} \delta_1(s) - \sum_{s \in \zeta_{k-1}^1} \bar{R}(s), \quad 2 \leq k \leq r_1 + 1$$

and  $\mu_1(C) = \gamma_{k+1}^1$  if  $C \neq T_1$  and  $|C| = \sum_{l=1}^k |\zeta_l^1|$ . This completes  $\mu_1$ .

Now we must construct  $\tau_1$ . Fix a strategy  $s \in S^1$ . For each set  $C \neq T_1$  such that  $s \in C$ ,  $s \succeq s'$  for all  $s' \in C$  and  $C \neq \bigcup_{s' \preceq s} s'$ , define

$$\tau_{1|C}(s) = \frac{1}{|\zeta(s) \cap C|}$$

For the set  $C = \bigcup_{s' \preceq s} s'$ , define

$$\tau_{1|C}(s) = \frac{\delta_1(s) - \bar{R}(s)}{\mu_1(C)}$$

Finally, set

$$\tau_{1|T}(s) = \frac{\sigma_1(s) - \delta_1(s)}{\mu_1(T)}$$

where  $\mu_1(T) = 1 - \sum_{C \neq T_1} \mu_1(C)$ . Note that in order for this construction to work, it must be that

$$\bar{R}(s) \leq \delta_1(s)$$

for each strategy  $s$ . Suppose  $s \in \zeta_1^1$ , then by the definition of  $\gamma_1^1$  one has that

$$\bar{R}(s) = \gamma_1^1 R(s) \leq \frac{\delta_1(s)}{R(s)} R(s) = \delta_1(s)$$

If  $s \in \zeta_k^1$  for  $k \geq 2$ , then

$$\bar{R}(s) = \sum_{\{C \subset \bigcup_{\{s' \preceq s\}} s', C \neq T_1 : s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta(s)|} \leq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}} 2^{|S^1|} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_{k-1}^1|-1}{z}}} 2^{|S^1|}$$

Using the fact that for all  $s \in \zeta_k^1$  and  $s' \in \zeta_{k-1}^1$ ,

$$\delta_1(s) \epsilon \geq \delta_1(s')$$

it must be that

$$\bar{R}(s) \leq \frac{m \delta_1(s) \epsilon}{\epsilon^{\frac{m}{z}}} 2^{|S^1|} = m \delta_1(s) 2^{|S^1|} \epsilon^{\frac{z-m}{z}} = m \delta_1(s) 2^{|S^1|} \epsilon^{\frac{1}{2}} \leq m \delta_1(s) 2^{|S^1|} \epsilon^{*\frac{1}{2}} = \frac{m \delta_1(s) 2^{|S^1|}}{m 2^m} \leq \delta_1(s)$$

There are still some details to verify. In particular, to show that  $\tau_1^{\mu_1} = \sigma_1$ , and that  $\mu_1$  is  $\epsilon^{\frac{1}{z}}$ -proper. For a strategy  $s$ , its probability under this specification is given by

$$\tau_1^{\mu_1}(s) = \bar{R}(s) + \delta_1(s) - \bar{R}(s) + \sigma_1(s) - \delta_1(s) = \sigma_1(s)$$

To verify properness, it suffices to show that for all  $k \geq 2$

$$\gamma_{k+1}^1 \epsilon^{\frac{1}{z}} \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}}$$

or

$$\gamma_{k+1}^1 \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}}$$

One has that

$$\gamma_{k+1}^1 = \sum_{s \in \zeta_k^1} (\delta_1(s) - \bar{R}(s)) \geq \sum_{s \in \zeta_k^1} \delta_1(s) - \frac{2^{|S^1|} - 1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \sum_{s' \in \zeta_{k-1}^1} \delta_1(s')$$

Furthermore

$$\frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_{k-1}^1|}{z}}}$$

Finally,

$$\sum_{s \in \zeta_k^1} \delta_1(s) \geq \frac{\max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{z-|\zeta_k^1|}{z}} \epsilon^{\frac{|\zeta_k^1|}{z}}} \geq \frac{\max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \frac{1}{\epsilon^{*\frac{1}{2}}} \geq \frac{\max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} m 2^{|S_1|} \geq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} 2^{|S_1|}$$

as desired. As a final note,  $\gamma_1^1$  was constructed so that properness holds for  $k = 1$ .

- Case 2:  $T = \bigcup_{l \leq k} \zeta_l^1$  for some  $k$ . It will be helpful to identify  $k_{T_1} = \max\{k : T_1 \cap \zeta_k^1 \neq \emptyset\}$ . For all  $k < k_{T_1} - 1$ , the construction is identical to Case 1. If  $|\zeta_{k_{T_1}}^1| \geq 2$ , then  $k_{T_1} - 1$  can also be constructed in the same way as Case 1. To handle instances in which  $|\zeta_{k_{T_1}}^1| \geq 2$ , one needs only to modify the construction from Case 1 for the sets

$$\mathbf{C}_{k_{T_1}} = \left\{ C : C \subset \bigcup_{l \leq k_{T_1}} \zeta_l^1, |C| = -1 + \sum_{l \leq k_{T_1}} |\zeta_l^1| \right\}$$

This modification is due to the fact that it is no longer possible to use the set

$$\bigcup_{l \leq k_{T_1}} \zeta_k^1 = T$$

to manage residual probabilities. Instead, these remaining probabilities must be distributed among the blocks in  $\mathbf{C}_{k_{T_1}}$ . To formalize this idea, define a modified version of  $\bar{R}$  for strategies in  $\zeta_{T_1}^1$ . For  $s \in \zeta_{k_{T_1}}^1$ , let

$$\bar{R}_{k_{T_1}}(s) = \sum_{\{C \subset \cup_{s' \leq s} s', |C| \leq -2 + \sum_{l \leq k_{T_1}} |\zeta_l^1|, s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta_{k_{T_1}}^1|}$$

where  $\mu_1(C)$  is constructed in the same way as Case 1 above for all sets  $|C| \leq -2 + \sum_{l \leq k_{T_1}} |\zeta_l^1|$

by using the  $\gamma_l^1$ . Now partition  $\mathbf{C}_{k_{T_1}}$  into two separate subsets.

$$\mathbf{C}_{k_{T_1}}^* = \{C : C \in \mathbf{C}_{k_{T_1}}, \zeta_{k_{T_1}}^1 \subseteq C\}$$

$$\mathbf{C}_{k_{T_1}}^{**} = \{C : C \in \mathbf{C}_{k_{T_1}}, \zeta_{k_{T_1}}^1 \not\subseteq C\}$$

Note that each element of  $\mathbf{C}_{k_{T_1}}^{**}$  contains all strategies in  $\zeta_{k_{T_1}}^1$  except for one. The idea is to use the smallest possible probability, while maintaining properness, for elements in  $\mathbf{C}_{k_{T_1}}^*$ . Then use  $\mathbf{C}_{k_{T_1}}^{**}$  to sort the rest of the remaining probabilities out. In particular, let

$$\mu_1(C) = \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1| - 1}{z}}}$$

$$\tau_{1|C}(s) = \frac{1}{|\zeta_{k_{T_1}}^1|}$$

for all  $C \in \mathbf{C}_{k_{T_1}}^*$  and  $s \in \zeta_{k_{T_1}}^1$ . For elements of  $\mathbf{C}_{k_{T_1}}^{**}$ , there is always one strategy from  $\zeta_{k_{T_1}}^1$  that is excluded. Enumerate the elements of  $\zeta_{k_{T_1}}^1$  as  $s_1, s_2, \dots, s_{|\zeta_{k_{T_1}}^1|}$ . Given  $C$ , suppose that  $s_i \notin C$ ,

then define

$$\mu_1(C) = \delta_1(s_{i+1}) - \bar{R}_{k_{T_1}}(s_{i+1}) - \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} \sum_{l \leq k_{T_1}-1} |\zeta_l^1|$$

$$\tau_{1|C}(s_{i+1}) = 1$$

where with an abuse of notation  $s_{|\zeta_{k_{T_1}}^1|+1} = s_1$ . Now to verify that

$$\mu_1(C) \geq \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}}$$

for all  $C \in \mathbf{C}_{k_{T_1}}^{**}$  in order to maintain properness. This is of course verified in the same way as Case 1. For any  $s \in \zeta_{k_{T_1}}^1$ ,

$$\delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} \sum_{l \leq k_{T_1}-1} |\zeta_l^1| \geq \delta_1(s) - \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} (2^{|S_1|} - 1)$$

Thus I would like to say that

$$\delta_1(s) \geq \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} 2^{|S_1|}$$

To conclude,

$$\frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} 2^{|S_1|} \leq \frac{\sum_{s' \in \zeta_{k_{T_1}-1}^1} \delta_1(s')}{\epsilon^{\frac{m}{z}}} 2^{|S_1|} \leq \frac{m \delta_1(s) \epsilon^*}{\epsilon^{*\frac{1}{2}}} 2^{|S_1|} = \delta_1(s)$$

Thus the distribution up to this point is  $\epsilon^{\frac{1}{z}}$  - *proper*. Define

$$\gamma_{k_{T_1}+1}^1 = \max_{s \in \zeta_{k_{T_1}}^1} \left[ \delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|^{-1}}{z}}} \sum_{l \leq k_{T_1}-1} |\zeta_l^1| \right] \frac{1}{\epsilon^{\frac{1}{z}}}$$

With  $\gamma_{k_{T_1}+1}^1$  defined, the rest of the construction is the same as Case 1, with the special note that

$$\mu_1(C) = \gamma_{k_{T_1}+1}^1 \epsilon^{\frac{1}{z}}$$

for all  $C$  such that  $C \notin \mathbf{C}_{k_{T_1}}$  with  $|C| = |T| - 1$ .

The sub-case in which  $|\zeta_{k_{T_1}}^1| = 1$  requires only slight modifications to the above argument.

It is immediate that  $\tau^{\mu_i}(s) = \sigma_i(s)$  for each player  $i$  and each strategy  $s \in S^i$ . Furthermore, since conditional distributions  $\tau_{i|C}$  are such that only best responses in  $C$  receive positive probability, the profile  $\tau$  constructed above is indeed an equilibrium of the consideration-set game.

□

### Proof of Lemma 5

PROOF. To show necessity, suppose that  $(p, q)$  is a  $T$ -equilibrium under perturbations  $\{\delta_1, \delta_2\}$ . Let  $I = Y_1^\delta(p) \subseteq T_1$  and  $J = Y_2^\delta(q) \subseteq T_2$ . Note that for all strategies outside of  $I$  ( $J$ ), it must be that  $p_i = \delta_1(s_i)$  ( $q_j = \delta_2(s_j)$ ). Finally note that  $I \subseteq BR_1^T(p, q)$  and  $J \subseteq BR_2^T(p, q)$ . Hence  $(p, \delta_1) \in S_{IJ}^1$  and  $(q, \delta_2) \in S_{IJ}^2$ .

To show sufficiency, suppose that  $(p, \delta_1) \in S_{IJ}^1$  and  $(q, \delta_2) \in S_{IJ}^2$  for some strategy sub-block  $I \times J$  of  $T$ . Clearly then  $Y_i^\delta(p, q) \subseteq BR_i^T(p, q)$  for each player  $i$ , and hence the tuple  $(p, q)$  is a  $T$ -equilibrium since the other constraints imposed by  $S_{IJ}^1$  and  $S_{IJ}^2$  ensure that these mixed strategies lie in the set of admissible strategies given the perturbation  $\delta$ .  $\square$

### Proof of Proposition 3

PROOF. I begin with the "only if" direction. Suppose that  $T$  is coarsely tenable. Suppose, seeking contradiction, there is some sub-block  $I \times J$  that possesses a solution to the above systems at  $\delta = 0$ , but fails at least one of (1) or (2). Fix this sub-block and suppose without loss of generality that (1) fails. Then  $\Omega_{IJ}^2$  is nonempty, and there is some  $s_\omega \in \Omega_{IJ}^2$  such that  $S_{IJ}^1$  possesses a vertex within the half-space  $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$ , call this vertex  $(p^*, \delta^*)$ . Furthermore, since  $s_\omega \in \Omega_{IJ}^2$ , the hyperplane  $\{(p, \delta) : pBe_\omega - pBe_j\}$  intersects  $S_{IJ}^1$  at a point where  $\delta_1 = 0$ . Hence there is some point  $(\hat{p}, 0) \in S_{IJ}^1$  and satisfies  $pBe_\omega = pBe_j$ . Using the fact that  $S_{IJ}^1$  is a convex set, it holds that  $\lambda(p^*, \delta^*) + (1 - \lambda)(\hat{p}, 0) \in S_{IJ}^1$  for all  $\lambda \in [0, 1]$ . Note also that

$$(\lambda p^* + (1 - \lambda)\hat{p})Be_\omega - (\lambda p^* + (1 - \lambda)\hat{p})Be_j > 0 \quad \forall \lambda \in (0, 1]$$

Therefore, for any  $\lambda \in (0, 1)$ , there is a tuple  $(\lambda p^* + (1 - \lambda)\hat{p}, \lambda \delta^*, q, 0)$  that solves the above system, and in which  $s_\omega$  is a strictly better reply to  $\lambda p^* + (1 - \lambda)\hat{p}$  than all elements of  $T_2$ . Sending  $\lambda \rightarrow 0$  and invoking Lemma 3 yields a contradiction.

Proceeding with the "if" direction, suppose that all sub-blocks outlined in the proposition satisfy (1) and (2). Suppose, seeking contradiction, that  $T$  is not coarsely tenable. Then there exists a sequence  $\epsilon_k \rightarrow 0$  and a sequence of (without loss of generality *simple*) distributions  $\mu_k$  with  $\mu_k(T) > 1 - \epsilon_k$  for each  $k$  such that there is an equilibrium  $\tau_k$  of the consideration-set game induced by  $\mu_k$  where some player, without loss of generality player 2, possesses a strategy  $s_\omega \in S_2 \setminus T_2$  such that

$$u_2(s_\omega, \tau_{1,k}^\mu) > \max_{t_2 \in T_2} u_2(t_2, \tau_{1,k}^\mu)$$

Consider the sequences  $\{\tau_{1|T_1}\}_k$  and  $\{\tau_{2|T_2}\}_k$  of conditional distributions over pure strategies in  $T$  at equilibrium. The support of  $\tau_{i|T_i}$  is a finite subset of  $T_i$  for each  $k$ . The Bolzano-Weierstrass theorem then implies that the sequence of supports possesses a convergent (constant) subsequence. Hence one can assume that the above sequence is constructed such that the supports of  $\{\tau_{i|T_i}\}_k$  remain constant for each  $i$ . Denote the support of  $\{\tau_{1|T_1}\}_k$  by  $I$ , and the support of  $\{\tau_{2|T_2}\}_k$  by  $J$ . Lemmas 2 and 5 imply that  $\tau_k$  corresponds to some  $(p, \delta_1) \in S_{IJ}^1$  and  $(q, \delta_2) \in S_{IJ}^2$  under the chosen sub-block of supports for each  $k$ . The fact that the equilibria of the consideration-set games converge to equilibria of the block game  $G^T$  as  $\mu(T) \rightarrow 1$  implies that the above sequence possesses a subsequence which converges to an equilibrium  $\tau^*$  of the block game. I suppose without loss of generality that the entire sequence has this property. This implies that  $S_{IJ}^1$  and  $S_{IJ}^2$  have solutions when  $\delta_i = 0$ . The fact that

$$u_2(s_\omega, \tau_{1,k}^\mu) > \max_{t_2 \in T_2} u_2(t_2, \tau_{1,k}^\mu)$$



for each  $k$  implies that the associated solution to the system is such that, for each  $k$ ,

$$p_k Be_\omega - p_k Be_j > 0$$

where  $p_k = \tau_{1,k}^\mu$  is the distribution over pure strategies for player 1 induced by  $\tau_k$ . Furthermore, sending  $\epsilon_k \rightarrow 0$  ( $\mu_k(T) \rightarrow 1$ ) implies that the hyperplane  $\{(p, \delta) : p Be_\omega - p Be_j = 0\}$  intersects the set of solutions to the system at a point in which  $\delta_1 = 0$ . Therefore,  $S_{IJ}^1$  has a vertex that lies within the half-space  $\{(p, \delta) : p Be_\omega - p Be_j > 0\}$ . This is a contradiction. Thus the result is proved.  $\square$

#### Proof of Proposition 4

PROOF. It suffices only to check sub-blocks  $I \times J$  in which both  $S_{IJ}^1$  and  $S_{IJ}^2$  possess solutions on the zero perturbation region. All other sub-blocks will of course lack  $I \times J$ -refutations since solutions of the corresponding system are bounded away from the zero perturbation region.

Suppose that no sub-block  $I \times J$  of  $T$  possesses an  $I \times J$ -refutation. Suppose, seeking contradiction, that  $T$  is not finely tenable. Then there exists a sequence  $\{\epsilon_k, \mu_k, \tau_k\}_{k=1}^\infty$  such that  $\epsilon_k \rightarrow 0$ ,  $\mu_k$  is an  $\epsilon_k$ -proper type distribution with respect to  $T$  and  $\tau_k$  is an equilibrium of the consideration-set game induced by  $\mu_k$  with the property that for each  $k$ , there is some player  $i$  and strategy  $s_\omega \in S_i \setminus T_i$  such that

$$u_i(s_\omega, \tau_{-i,k}) > \max_{t_i \in T_i} u_i(t_i, \tau_{-i,k})$$

The sequence can be chosen such that the above inequality holds for the same player and the same strategy  $s_\omega$  for each  $k$ . Furthermore, it is safe to assume that  $\text{supp}_i(\tau_{i|T_i})_k$  is constant for each player  $i$  and all  $k$ . These follow from application of Bolzano-Weierstrass. Let  $I = \text{supp}_1(\tau_{1|T_1})_k$  and  $J = \text{supp}_2(\tau_{2|T_2})_k$ . For each  $k$ , Lemma 4 implies that there is a  $T(\epsilon_k^{\frac{1}{3}})$ -proper tuple  $(\sigma_k, \delta_k)$  that solves the systems  $S_{IJ}^1$  and  $S_{IJ}^2$ . This implies that the sequence of tuples  $\{\epsilon_k^{\frac{1}{3}}, \sigma_k, \delta_k\}_{k=1}^\infty$  obtained from the projections of  $\tau_k^{\mu_k}$  constitutes an  $I \times J$ -refutation, a contradiction.

Proceeding with the "only if" direction, suppose that  $T$  is finely tenable. Suppose, seeking contradiction, that there is some sub-block  $I \times J$  of  $T$  which possesses an  $I \times J$ -refutation. Fix this sub-block and let  $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^\infty$  denote a refutation. Suppose without loss of generality that  $m = |S_1| = \max\{|S_1|, |S_2|\}$ . Since  $\epsilon_k \rightarrow 0$ , there is  $K \in \mathbb{N}$  such that for all  $k \geq K$ ,  $\epsilon_k \leq \frac{1}{m^2 2^{2m}} = \epsilon^*$ . Proposition 2 then implies that for all  $k \geq K$ , there is a type distribution  $\mu_k$  that is  $\epsilon_k^{\frac{1}{2m}}$ -proper and  $\tau_k$  such that  $\tau_k^{\mu_k} = \sigma_k$ . This immediately provides a sequence  $\{(\epsilon_k)^{\frac{1}{2m}}, \mu_k, \tau_k\}_{k \geq K}$  that constitutes a direct violation of fine tenability, a contradiction.  $\square$

#### Proof of Lemma 6

PROOF. The first condition is clearly necessary, as  $T(\epsilon)$ -proper tuples tend to zero perturbation at  $\epsilon \rightarrow 0$ .

Suppose, seeking contradiction, that  $I \times J$  contains  $T(\epsilon)$ -proper tuples  $(\sigma_1, \delta_1)$  according to the order  $\succeq_1$  for all  $\epsilon > 0$ , but that the second condition above fails. Then there is some strategy  $s^* \in S_1$  such that for every vertex  $v$  of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  of the form

$$\{v : v(\delta_1(s^*)) > 0\} := \text{Pos}(s^*)$$

there is some strategy  $s' \in S_1$  in which  $s > s'$  and  $v(\delta_1(s')) > 0$ . Let

$$\epsilon^* = \frac{\min_{v \in \text{Pos}(s^*)} \min_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))}{|\text{Pos}(s^*)| \max_{\{v \in \text{Pos}(s^*)\}} v(\delta_1(s^*))}$$

I claim that there is no tuple  $(\sigma_1, \delta_1) \in cl(S_{IJ}^1 \cap p(\succeq_2))$  that is  $T(\frac{\epsilon^*}{2})$  – *proper* according to the order  $\succeq_1$ . Let,

$$\{v : v(\delta_1(s^*)) = 0\} := Zero(s^*)$$

and consider any element

$$(\sigma_1, \delta_1) = \sum_{v \in Pos(s^*)} \lambda_v v + \sum_{v \in Zero(s^*)} \lambda_v v$$

within  $cl(S_{IJ}^1 \cap p(\succeq_2))$ . In order for properness to hold for any  $\epsilon$ , it must be that  $\delta_1(s) > 0$  for all  $s \in S_1$ . It must then be that there is  $v \in Pos(s^*)$  with  $\lambda_v > 0$ . Let  $\lambda_{Pos(s^*)}^* = \max_{\{v \in Pos(s^*)\}} \lambda_v$  and let  $v^*$  be a maximizing vertex. Choose any  $s' \prec_1 s$  such that  $v^*(\delta_1(s')) > 0$ . Then,

$$\begin{aligned} \frac{\delta_1(s')}{\delta_1(s^*)} &= \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s')) + \sum_{v \in Zero(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))} \geq \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))} \\ &\geq \frac{\lambda_{Pos(s^*)}^* \min_{v \in Pos(s^*)} \min_{\{s: v(\delta_1(s)) > 0\}} v(\delta_1(s))}{\lambda_{Pos(s^*)}^* |Pos(s^*)| \max_{\{v \in Pos(s^*)\}} v(\delta_1(s^*))} = \epsilon^* \end{aligned}$$

Hence there are no  $T(\frac{\epsilon^*}{2})$  – *proper* tuples in  $cl(S_{IJ}^1 \cap p(\succeq_2))$ , a contradiction.  $\square$

### Proof of Lemma 7

PROOF. The proof is a straightforward result of the fact that  $S_{IJ}^1$  and  $p(\succeq_2)$  are convex. This implies that their intersection is convex, and hence

$$relint(cl(S_{IJ}^1) \cap p(\succeq_2)) = relint(S_{IJ}^1 \cap p(\succeq_2)) \subseteq S_{IJ}^1 \cap p(\succeq_2)$$

where the first equality follows from known results about convex sets. (See Proposition 1.4.3 of [Bertsekas et al. 2003]).  $\square$

### Proof of Lemma 8

PROOF. Given  $\epsilon \in (0, 1)$ , I seek to construct a tuple  $(\sigma_1, \delta_1)$  that satisfies the properness criteria for  $\epsilon$  and such that

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S_{IJ}^1 \cap p(\succeq_2))\}} \lambda_v v$$

with  $\lambda_v > 0$  for all vertices  $v$ . Fix the following values:

$$\bar{\delta} = \max_{\{v \in cl(S_{IJ}^1 \cap p(\succeq_2))\}} \max_{\{s: v(\delta_1(s)) > 0\}} v(\delta_1(s))$$

$$\underline{\delta} = \min_{\{v \in cl(S_{IJ}^1 \cap p(\succeq_2))\}} \min_{\{s: v(\delta_1(s)) > 0\}} v(\delta_1(s))$$

Partition  $S^1$  into the indifference classes  $\zeta_1^1, \zeta_2^1, \dots, \zeta_{r_1}^1$  according to  $\succeq_1$ , but in decreasing order this time (i.e.  $\zeta_1^1$  contains the best strategies according to  $\succeq_1$ ). For each  $\zeta_k^1$  and each  $s \in \zeta_k^1$ , the conditions of Lemma 6 imply that there is a corresponding vertex  $v_s$  such that  $v_s(\delta_1(s)) > 0$  and for each  $\zeta_j^1$  with  $j > k$ ,

$$v_s(\delta_1(s')) = 0 \quad \forall s' \in \zeta_j^1$$

Note that multiple strategies within the same indifference class may possess the same corresponding vertex. For each indifference class  $\zeta_k^1$ , choose a collection of vertices  $V_k$  such that for each strategy  $s \in \zeta_k^1$ , there is a vertex  $v_s \in V_k$  such that  $v(\delta_1(s)) > 0$  and for each  $\zeta_j^1$  with  $j > k$ ,

$$v_s(\delta_1(s')) = 0 \quad \forall s' \in \zeta_j^1$$

Note that  $V_k \cap V_j = \emptyset$  for  $k \neq j$ . For  $v \in V_k$ , let

$$\lambda_v = \frac{(\epsilon \bar{\delta})^k}{(|S^1|d)^k}$$

where  $d$  is the number of vertices of  $cl(S_{IJ}^1 \cap p(\succeq_2))$ . For all vertices  $v \notin \bigcup_k V_k$  that do not lie on the zero perturbation region, let

$$\lambda_v = \frac{(\epsilon \bar{\delta})^{r_1}}{(|S^1|d)^{r_1}}$$

For vertices on the zero perturbation region (of which there is at least one by assumption), allocate all remaining probability uniformly. Note that under this construction all  $\lambda_v$  can be scaled down in order to construct a valid probability distribution while maintaining the requisite properness criteria. Let us now verify that the profile

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S_{IJ}^1 \cap p(\succeq_2))\}} \lambda_v v$$

is  $T(\epsilon)$ -proper. The fact that  $(\sigma_1, \delta_1) \in S_{IJ}^1$  (Lemma 7) implies that the first two conditions are satisfied. The third condition is clearly satisfied. To verify the fourth condition, let  $s \in \zeta_k^1$  and  $s' \in \zeta_{k+1}^1$  for  $k \leq r_1 - 1$ . Then

$$\begin{aligned} \delta_1(s) &\geq \frac{(\epsilon \bar{\delta})^k}{(|S^1|d)^k} \\ \delta_1(s') &\leq \bar{\delta} \left( d - \sum_{i=1}^{r_1} |V_i| \right) \frac{(\epsilon \bar{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon \bar{\delta})^i \bar{\delta}}{(|S^1|d)^i} \end{aligned}$$

then

$$\begin{aligned} \frac{\delta_1(s')}{\delta_1(s)} &\leq \frac{\bar{\delta} \left( d - \sum_{i=1}^{r_1} |V_i| \right) \frac{(\epsilon \bar{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon \bar{\delta})^i \bar{\delta}}{(|S^1|d)^i}}{\frac{(\epsilon \bar{\delta})^k}{(|S^1|d)^k}} \\ &\leq \frac{d \bar{\delta} (\epsilon \bar{\delta})^{k+1}}{(\epsilon \bar{\delta})^k} \frac{(|S^1|d)^k}{(|S^1|d)^{k+1}} = \frac{\bar{\delta} \bar{\delta} \epsilon}{|S^1|} \leq \epsilon \end{aligned}$$

The final condition for  $(\sigma_1, \delta_1)$  to be  $T(\epsilon)$ -proper can be made to hold due to the fact that all  $\lambda_v$  that do not lie on the zero perturbation region may be scaled down while maintaining the first four conditions. Hence it is possible to extract a profile  $(\sigma_1, \delta_1) \in relint(cl(S_{IJ}^1) \cap p(\succeq_2))$  which is  $T(\epsilon)$ -proper when the conditions of Lemma 6 are satisfied.  $\square$

### Proof of Proposition 5

PROOF. Suppose that  $I \times J$  possesses a refutation  $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^\infty$  with constant total preorders  $\{\succeq_1, \succeq_2\}$ . One must show that

- (1) The vertices of  $cl(S_{IJ}^1 \cap p(\succeq_2))$  and  $cl(S_{IJ}^2 \cap q(\succeq_1))$  are also vertices of  $H_{IJ}^1$  and  $H_{IJ}^2$  respectively.
- (2) The total preorders  $\succeq_1$  and  $\succeq_2$  satisfy the restrictions imposed by steps 2 – 5 of Algorithm 1.
- (3) The total preorders  $\succeq_1$  and  $\succeq_2$  satisfy the conditions of Lemma 6.

To verify (1), note that  $cl(S_{IJ}^1 \cap p(\succeq_2))$  corresponds to a face of the arrangement  $H_{IJ}^1$ . Similarly  $cl(S_{IJ}^2 \cap q(\succeq_1))$  is a face of  $H_{IJ}^2$ .

Condition (2) must clearly be satisfied as a refutation profile  $\sigma_k$  with constant total preorders for all  $k$  has a subsequence which converges to some point on  $Z_{IJ}^1 \times Z_{IJ}^2 = Conv(V_Z^1) \times Conv(V_Z^2)$  as  $\epsilon_k \rightarrow 0$ .

Lemmas 6-8 guarantee that if such  $(\sigma_k, \delta_k)$  exists then the sets  $cl(S_{IJ}^1 \cap p(\succeq_2))$  and  $cl(S_{IJ}^2 \cap q(\succeq_1))$  satisfy the conditions of Lemma 6. Thus Algorithm 1 detects the refutation  $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^\infty$  provided it compares the preorders  $\{\succeq_1, \succeq_2\}$  associated with the sequence  $\sigma_k$ .

On the other hand, if the algorithm encounters two total preorders  $\succeq_1$  and  $\succeq_2$  such that  $cl(S_{IJ}^1 \cap p(\succeq_2))$  and  $cl(S_{IJ}^2 \cap q(\succeq_1))$  satisfy the conditions of Lemma 6, then Lemma 6 implies that it is possible to construct an  $I \times J$ -refutation.  $\square$

### Proof of Lemma 9

PROOF. Begin by proving the first claim of the lemma. Suppose seeking contradiction and w.l.o.g. that player one possesses a pair of variables  $x_j$  and  $x_k$  such that

$$\sigma_1(+l_j) + \sigma_1(-l_j) > \sigma_1(+l_k) + \sigma_1(-l_k)$$

Then it must be that

$$u_2(\sigma_1, x_j) > u_2(\sigma_1, x_k)$$

Thus  $x_j \succ_2 x_k$  and  $\sigma_2(x_j)\epsilon \geq \sigma_2(x_k)$ . Note that as  $\epsilon \rightarrow 0$  one has that

$$x \succ_2 l \quad \forall x \in V, l \in L$$

and so

$$\sigma_2(x_j)\epsilon^2 \geq \sigma_2(x_k)\epsilon \geq \sigma_2(l) \quad \forall l \in L$$

As  $\epsilon \rightarrow 0$ , it must then be that

$$u_1(\pm l_j, \sigma_2) < u_1(\pm l_k, \sigma_2)$$

and thus  $\sigma_1$  does not satisfy criteria for properness.

Similarly, suppose that  $\sigma_1(x_j) > \sigma_1(x_k)$ . Then  $\pm l_k \succ_2 \pm l_j$ , and hence player two places much less probability on  $\{+l_j, -l_j\}$  than on  $\{+l_k, -l_k\}$  as  $\epsilon \rightarrow 0$ . This implies that  $u_1(x_j, \sigma_2) < u_1(x_k, \sigma_2)$ , and hence  $\sigma_1$  cannot be proper.  $\square$

### Proof of Lemma 10

PROOF. Suppose without loss of generality and seeking contradiction that  $\sigma_1$  is such that there exists two variables  $x_i$  and  $x_j$  with  $\max\{\sigma_1(+l_i), \sigma_1(-l_i)\} > \max\{\sigma_1(+l_j), \sigma_1(-l_j)\}$ . Lemma 9 of course implies that  $\min\{\sigma_1(+l_i), \sigma_1(-l_i)\} < \min\{\sigma_1(+l_j), \sigma_1(-l_j)\}$ . This implies that

$$\max\{u_2(\sigma_1, +l_i), u_2(\sigma_1, -l_i)\} > \max\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\} \geq \min\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\}$$

and hence  $\sigma_2(+l_i) + \sigma_2(-l_i) > \sigma_2(+l_j) + \sigma_2(-l_j)$  in order to maintain properness. But this violates Lemma 9, a contradiction. Thus for every pair of variables  $x_i$  and  $x_j$

$$\max\{\sigma_i(+l_i), \sigma_i(-l_i)\} = \max\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

$$\min\{\sigma_i(+l_i), \sigma_i(-l_i)\} = \min\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

for each player  $i$ . Thus a partition  $L_1^i$  and  $L_2^i$  exists for individual players. To show that  $L_1^1 = L_1^2$ , note that if  $\sigma_i(+l_j) > \sigma_i(-l_j)$  then  $u_{-i}(\sigma_{-i}, +l_j) > u_{-i}(\sigma_{-i}, -l_j)$  and hence  $\sigma_{-i}(+l_j) > \sigma_{-i}(-l_j)$ . Similarly

it is easy to prove that if  $\sigma_i(+l_j) = \sigma_i(-l_j)$ , it must be that  $\sigma_{-i}(+l_j) = \sigma_{-i}(-l_j)$ . Thus it must be that  $L_1^1 = L_1^2$  and  $L_2^1 = L_2^2$ .  $\square$

### Proof of Lemma 11

PROOF. Let  $A = \{l_1, l_2, \dots, l_n\}$  denote a satisfying assignment of  $\phi$ . Let  $C_1 = \{c \in C : c \cap A = 1\}$ ,  $C_2 = \{c \in C : c \cap A = 2\}$  and  $C_3 = \{c \in C : c \cap A = 3\}$ . Construct a refutation as follows:

$$\begin{aligned} \delta_i(f) &= \sigma_i(f) = \epsilon \\ \delta_i(c) &= \sigma_i(c) = \epsilon^2 \quad \forall c \in C_1 \\ \delta_i(c) &= \sigma_i(c) = \epsilon^3 \quad \forall c \in C_2 \\ \delta_i(c) &= \sigma_i(c) = \epsilon^4 \quad \forall c \in C_3 \\ \delta_i(g) &= \epsilon^5 \\ \delta_i(x) &= \sigma_i(x) = \epsilon^6 \quad \forall x \in V \\ \delta_i(l) &= \sigma_i(l) = \epsilon^7 \quad \forall l \in A \\ \delta_i(h) &= \sigma_i(h) = \epsilon^8 \\ \delta_i(l) &= \sigma_i(l) = \epsilon^9 \quad \forall l \notin A \\ \sigma_i(g) &= 1 - \sum_{s \neq g} \sigma_i(s) \end{aligned}$$

I have omitted the normalizing factor that would ensure that  $\sigma_i(g) > 1 - \epsilon$ , but this does not present an issue to the proof. All probabilities aside from  $\sigma_i(g)$  could be scaled down by for instance a factor of  $\frac{1}{2}$  to ensure this holds. Now to show that it is indeed the case that

$$f >_i c >_i g > x >_i l (\in A) >_i h >_i l (\notin A)$$

for each player  $i$ . Note that since  $\sigma_i(g) \rightarrow 1$  as  $\epsilon \rightarrow 0$ , it must be that  $f, g, c > x > l, h$  for small  $\epsilon$ . Computing expected payoffs:

$$\begin{aligned} u_i(\sigma_{-i}, f) &= \left(\frac{n-\frac{1}{2}}{n}\right)(n\epsilon^7 + n\epsilon^9) + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \\ u_i(\sigma_{-i}, c) &= (n-1)\epsilon^7 + (n-2)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_1 \\ u_i(\sigma_{-i}, c) &= (n-2)\epsilon^7 + (n-1)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_2 \\ u_i(\sigma_{-i}, c) &= (n-3)\epsilon^7 + (n)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_3 \\ u_i(\sigma_{-i}, g) &= n(|C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4) + (1 - \epsilon - |C_1|\epsilon^2 - |C_2|\epsilon^3 - |C_3|\epsilon^4)(n-1) \\ u_i(\sigma_{-i}, l) &= (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + \\ &\quad n(n-1)\epsilon^7 + (n-1)^2\epsilon^9 + (n-4)\epsilon^9 + (n-3)\sigma_i(g) \quad \forall l \in A \\ u_i(\sigma_{-i}, h) &= (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + \\ &\quad n\left(n-1-\frac{1}{2n}\right)(\epsilon^7 + \epsilon^9) + (n-3)\sigma_i(g) \\ u_i(\sigma_{-i}, l) &= (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + \\ &\quad (n-1)^2\epsilon^7 + (n-4)\epsilon^7 + n(n-1)\epsilon^9 + (n-3)\sigma_i(g) \quad \forall l \notin A \end{aligned}$$

Using these calculated payoffs, it is possible to verify that the appropriate ordering holds for  $\succeq_i$ , and hence this profile serves as a refutation.  $\square$

### Proof of Lemma 12

PROOF. Suppose that  $T$  has a refutation. Then the total preorder must satisfy  $f \succeq_i c$  for all clauses  $c$  for  $\epsilon$  sufficiently small. Indeed, if  $c^* \succ_i f$  for some clause  $c^*$ , then  $g \succ_i c^*$  and hence the preorder can't serve as a refutation for all  $\epsilon$  as  $g$  would then be the most preferred strategy. Lemmas 9 and 10 also imply that, for  $\epsilon$  small,  $f \sim_i c$  cannot occur. This is due to the fact that the payoffs from  $f$  and  $c$  differ only on literals and  $h$ . In particular, let  $L_1$  and  $L_2$  be a partition from Lemma 10 and suppose that  $L_1 \succ_i L_2$  (this implies that  $L_1 \succ_i h$ ). Noting that each literal in  $L_1$  must be given the same probability at  $T(\epsilon)$  proper profiles, we see that for all sufficiently small  $\epsilon$ , either  $c \succ_i f$  or  $f \succ_i c$ . If it is the case that  $L_1 \sim_i L_2$ , then  $h \succ_i L_1, L_2$  so that  $c \succ_i f$  for all clauses  $c$ . Hence there cannot be a refutation due to the fact that  $g \succ_i c$ . Therefore, if a refutation exists, the total preorder must satisfy  $f \succ_i c$  for all clauses  $c$ , and  $L_1 \succ L_2$  in the partition from Lemma 10. Note that if  $L_1 \succ_i L_2$ , it must be the case that  $h$  lies between them so that  $L_1 \succ_i h \succ_i L_2$ . Given this, if  $f \succ_i c$  for all clauses, then every clause  $c$  possesses a literal from the set  $L_1$ . To see why, if some clause  $c^*$  is such that every literal from  $L_1$  is not in  $c^*$ , then it must be the case that  $c \succ_i f$ , and hence  $g \succ_i f$ . Thus if a refutation exists, then it must be the case that there is an assignment of literals  $L_1$  such that each clause possesses a literal from  $L_1$ . Hence  $L_1$  serves as a satisfying assignment.  $\square$

### Proof of Lemma 13

PROOF. Suppose that one is given a sub-block  $I \times J$  of  $T$  and a pair of total preorders  $\{\succeq_1, \succeq_2\}$ . To verify whether they serve as a refutation, it suffices to show that  $p(\succeq_2)$  contains profiles which are  $T(\epsilon)$ -proper in its relative interior, with a similar statement holding for  $q(\succeq_1)$  with  $\epsilon$  sufficiently small. Lemmas 6-8 imply that one need only determine whether there are a collection of vertices of  $p(\succeq_2)$  and  $q(\succeq_1)$  that satisfy the conditions of lemma 6. Consider w.l.o.g. the total preorder  $\succeq_1$  and enumerate the indifference classes  $\zeta_1^1, \dots, \zeta_r^1$  in decreasing order, so that strategies in  $\zeta_i^1$  are strictly preferred to all  $s' \in \zeta_j^1$  with  $j > i$ . To verify the conditions of lemma 6 for a given strategy  $s \in \zeta_i^1$ , it suffices to solve

$$\begin{aligned} & \max \delta_1(s) \\ \text{s.t. } & \delta_1(s') = 0 \quad \forall s' \in \zeta_j^1, j > i \\ & \delta_1(s) \in \text{cl}(S_{IJ}^1 \cap p(\succeq_2)) \end{aligned}$$

and verify whether there is a basic feasible solution with  $\delta_1(s) > 0$ . Note that the constraint set for this problem is defined by a number of linear inequalities that is polynomial in  $|S_1|$  and  $|S_2|$ . If the desired solution exists for all  $s \in S_1$ , then lemma 8 implies that there are  $T(\epsilon)$ -proper profiles according to  $\succeq_1$  that lie within  $p(\succeq_2)$ . A similar process can be performed for strategies in  $\succeq_2$ .  $\square$

### Proof of Proposition 7

PROOF. By proposition 4, the problem of verifying whether a given block  $T$  is finely tenable is equivalent to determining whether there is a sub-block  $I \times J$  that possesses an  $I \times J$ -refutation. Lemmas 6-8 imply that any  $I \times J$ -refutation can be equivalently expressed as a pair of total preorders  $\{\succeq_1, \succeq_2\}$ . Thus the problem of verifying fine tenability of a block  $T$  is equivalent to finding  $\{I, J, \succeq_1, \succeq_2\}$  with  $I \times J$  a sub-block of  $T$  and  $\succeq_1, \succeq_2$  satisfying the conditions outlined in lemma 6. Given such a tuple  $\{I, J, \succeq_1, \succeq_2\}$ , lemma 13 implies that it can be verified in polynomial time. This, combined with proposition 7, implies that the problem is NP-complete.  $\square$

### Proof of Lemma 14

PROOF. To prove necessity, suppose that  $\sigma = (p^*, q^*)$  is indeed a proper equilibrium. Then there is a sequence  $\{\epsilon_k, \sigma_k\}_{k=1}^{\infty}$  of  $\epsilon_k$ -proper equilibria converging to  $\sigma$  with  $\epsilon_k \rightarrow 0$ . The total preorders over strategies induced by  $\sigma_{1,k}$  and  $\sigma_{2,k}$  can be taken to be constant in  $k$ . Hence the sequence

$\sigma_{1,k}$  and  $\sigma_{2,k}$  lie on single closed faces  $f_1$  and  $f_2$  respectively. Since these faces are closed and the sequences converge to  $p^*$  and  $q^*$ , it must be that  $p^* \in f_1$  and  $q^* \in f_2$ . The fact that  $\sigma_k$  are  $\epsilon_k$ -proper with  $\epsilon_k \rightarrow 0$  implies that  $f_i \subseteq P^i(p^*, q^*)$ . Showing the final condition is almost identical to the logic of Lemma 6.

To show sufficiency, it suffices to construct a convergent sequence of  $\epsilon_k$ -proper equilibria directly. Since  $p^* \in f_1$ , it follows that for every strategy  $s \in Y_1(p^*)$ , there is a vertex  $v$  of  $f_1$  in which  $v(p_s) > 0$  and in which  $v(p_{s'}) = 0$  for all  $s' \notin Y_1(p^*)$ . Furthermore,  $p^*$  can be realized as a convex combination of such vertices. Let  $V_{p^*}$  denote the largest collection of vertices of  $f_1$  such that  $v(p_s) > 0$  for some  $s \in Y_1(p^*)$  and  $v(p_{s'}) = 0$  for all  $s' \notin Y_1(p^*)$ , and let  $\{\lambda_v^*\}_{v \in V_{p^*}}$  denote a convex combination such that

$$\sum_{v \in V_{p^*}} \lambda_v^* v = p^*$$

Let  $Z(V_{p^*}) = \{v \in V_{p^*} : \lambda_v^* = 0\}$ . Fix the following values:

$$\bar{p} = \max_{v \in f_1} \max_{\{s: v(p_s) > 0\}} v(p_s)$$

$$\underline{p} = \min_{v \in f_1} \min_{\{s: v(p_s) > 0\}} v(p_s)$$

Let  $\succeq_1$  denote the total preorder induced over  $S_1$  induced by strategies on the relative interior of  $f_2$ . Let  $V_1$  denote the collection of vertices of  $f_1$  such that  $v(p_s) > 0$  for some  $s \in \zeta_1^1 \setminus Y_1(p^*)$  and  $v(p_{s'}) = 0$  for all  $s' \in \zeta_k^1$ ,  $k > 1$ . Similarly, let  $V_k$ ,  $k \geq 2$ , denote the collection of vertices of  $f_1$  such that  $v(p_s) > 0$  for some  $s \in \zeta_k^1$  and  $v(p_{s'}) = 0$  for all  $s' \in \zeta_l^1$ , with  $l > k$ . Note that  $V_k \cap V_j = \emptyset$  for  $j \neq k$ . Let  $|V(f_1)|$  denote the total number of vertices of the closed face  $f_1$ . For a given, small  $\epsilon > 0$ , I will construct a profile  $p$  that lies within the relative interior of  $f_1$  and is  $\epsilon$ -proper according to  $\succeq_1$ . For  $v \in V_k$ , define

$$\lambda_v = \frac{(\epsilon \underline{p})^k}{(|S_1| |V(f_1)|)^k}$$

For  $v \notin \{\cup_k V_k\} \cup V_{p^*}$ , define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1| |V(f_1)|)^{r_1}}$$

For  $v \in Z(V_{p^*})$ , define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1| |V(f_1)|)^{r_1}}$$

For  $v \in V_{p^*} \setminus Z(V_{p^*})$ , define

$$\lambda_v = (1 - \sum_{\{v' \notin V_{p^*} \setminus Z(V_{p^*})\}} \lambda_{v'}) \lambda_v^*$$

Note that as  $\epsilon \rightarrow 0$ ,

$$\sum_{v' \notin V_{p^*} \setminus Z(V_{p^*})} \lambda_{v'} \rightarrow 0$$

Therefore

$$p_\epsilon = \sum_{v \in f_1} \lambda_v v \rightarrow p^*$$

as  $\epsilon \rightarrow 0$  since  $p^* = \sum_{v \in V_{p^*}} \lambda_v^* v$ . Note also that  $p_\epsilon(s) > 0$  for all  $s \in S_1$  and all  $\epsilon > 0$ . All that is left to show is that  $p_\epsilon$  is indeed  $\epsilon$ -proper according to  $\succeq_1$  for small enough  $\epsilon$ . For any  $s \in \zeta_k^1 \setminus Y_1(p^*)$ , with  $1 \leq k \leq r_1 - 1$ ,

$$p_\epsilon(s) \geq \frac{(\epsilon \underline{p})^k}{(|S_1| |V(f_1)|)^k}$$

and for any  $s' \in \zeta_{k+1}^1$

$$p_\epsilon(s') \leq \sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1| |V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1| |V(f_1)|)^{r_1}}$$

and hence

$$\begin{aligned} \frac{p_\epsilon(s')}{p_\epsilon(s)} &\leq \frac{\sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1| |V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1| |V(f_1)|)^{r_1}}}{\frac{(\epsilon \underline{p})^k}{(|S_1| |V(f_1)|)^k}} \\ &\leq \frac{|V(f_1)| \bar{p} (\epsilon \underline{p})^{k+1}}{(|S_1| |V(f_1)|)^{k+1}} \frac{(|S_1| |V(f_1)|)^k}{(\epsilon \underline{p})^k} = \frac{\bar{p} (\epsilon \underline{p})}{|S_1|} \leq \epsilon \end{aligned}$$

Now let  $s \in Y_1(p^*)$  and  $s' \in \zeta_2^1$ . Then

$$\begin{aligned} p(s) &\geq \underline{p} \min_{\{v: \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \setminus Z(V_{p^*})\}} \lambda_{v'}^*) \\ p(s') &\leq \sum_{i=2}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1| |V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1| |V(f_1)|)^{r_1}} \end{aligned}$$

Then

$$\frac{p(s')}{p(s)} \leq \frac{\bar{p} \epsilon^2}{|S_1|^2 |V(f_1)| \min_{\{v: \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \setminus Z(V_{p^*})\}} \lambda_{v'}^*)} \leq \epsilon$$

for  $\epsilon$  small enough. Thus  $p_\epsilon$  constructed here is a sequence of  $\epsilon$ -proper equilibria which converge to  $p^*$  as  $\epsilon \rightarrow 0$ . Note that since  $p_\epsilon$  are all in the relative interior of  $f_1$ , they induce the same total preorder over  $S_2$ . A mirrored construction allows one to construct  $q_\epsilon$  which are  $\epsilon$ -proper for  $\epsilon$  small enough and which lies in the relative interior of  $f_2$ , and hence induce the same total preorder over strategies in  $S_1$ . This provides a pair  $(p_\epsilon, q_\epsilon)$  satisfying the desired properties for  $\epsilon$  small and converging to  $(p^*, q^*)$  as  $\epsilon \rightarrow 0$ .  $\square$