On the Characteristics and Verification of Tenable Strategy Sets in Bimatrix Games

TYLER PORTER

I consider the robust set-valued equilibrium concepts *coarse tenability* and *fine tenability* introduced in [Myerson and Weibull 2015]. I show how their framework of *consideration-set* games maps onto the more familiar framework of strategy perturbations. This allows me to compare these set-valued concepts to other objects from the equilibrium refinement literature, as well as provide methods to verify whether strategy sets satisfy these robustness properties in bimatrix games. I provide complexity results for the verification of *fine tenability* and show how the methods developed for verifying this concept can be applied to proper equilibria.

ACKNOWLEDGEMENTS

I thank Jörgen Weibull for assistance with understanding the block solution concepts. David Easley provided a number of helpful discussions on the direction of the paper. Eva Tardos and Larry Blume provided assistance with technical aspects and organization of the paper. I would also like to thank four anonymous referees at EC'20. This paper would not be complete without their help.

1 INTRODUCTION

Since the seminal works of [von Neumann and Morgenstern 1944] and [Nash 1951], game theoretic methods have been brought to bear on problems of strategic interaction of every variety. The need for solution concepts with more cutting power than Nash equilibrium has led to the vast literature on equilibrium refinements. Many of these refinements come with different levels of agent sophistication or particular game forms in mind. In this paper, I study the characteristics and provide methods for the verification of the robust set-valued equilibrium concepts introduced in [Myerson and Weibull 2015]. These concepts are motivated by, on the one hand, models of evolutionary game theory in which a population of players is called upon to engage in strategic interaction with norms and historical precedent playing a role; and, on the other hand models of rationalizable choice and inattention. A central idea is that within a population a particular convention arises in which only a certain collection of strategies for players in various type roles are ever played. A natural question then is which conventions survive the test of time? In order to answer this question, [Myerson and Weibull 2015] discuss multiple solution concepts that describe when a convention is robust against small deviations from convention within the population. This paper is part of the growing literature on equilibrium refinement computation. I do not attempt to survey the entirety of this literature here, but I will briefly discuss those works which are most related to this one.

Few papers deal with the computation or verification of set-valued equilibrium concepts. [Benisch et al. 2010] provide methods to compute the minimal sets which are closed under rational behavior (CURB) of [Basu and Weibull 1991] in polynomial time by building such sets upon individual strategies. For finite games of 3 or more players, it was shown in [Klimm et al. 2011] that the problem of computing a minimal CURB set as well as the problem of verifying whether a given set is a minimal CURB set are both NP-complete. In this same paper, it is shown that computing strong CURB sets (in which correlated strategies are allowed) can be done in polynomial time by modifying the methods of [Benisch et al. 2010]. [Brandt and Brill 2016] discuss the computation of several set-valued solution concepts which can be expressed using the dominance structures defined in [Duggan and Breton 2014]. [Jansen and Vermeulen 2001] provide a framework for verifying whether a collection of equilibria satisfy *stability* as defined by [Kohlberg and Mertens 1986]. I adapt their framework in Section 4 below for the current problem.

Manuscript submitted for review to the 21st ACM Conference on Economics & Computation (EC'20).

Much of the literature on the computation of equilibria and equilibrium refinements focuses on point-valued concepts. The problem of computing a Nash equilibrium in a bimatrix game is known to be PPAD-complete due to a series of results by [Chen and Deng 2006] and [Daskalakis et al. 2006]. [Conitzer and Sandholm 2008] extend the results of [Gilboa and Zemel 1989] to show that many problems such as approximating the maximum social welfare at any Nash equilibrium are hard even for symmetric bimatrix games. For papers discussing the complexity and computation of equilibrium refinements in finite games see [Miltersen and Sørensen 2010], [Hansen et al. 2010], [Hansen 2017] and references therein.

It is important to note that the complexity and computation of proper equilibria is particularly relevant to the discussion in this paper. The concept of *fine tenability*, formally defined later, is quite similar to proper equilibrium in that it imposes a rationality requirement on trembles. I show that the exact methods I develop for fine tenability can also be applied to the verification of proper equilibria in bimatrix games. [Hansen and Lund 2018] show that the complexity of verifying the conditions imposed by proper equilibrium is NP-complete in bimatrix games. I show a similar result for verifying the conditions of fine tenability by providing a nontrivial modification of their reduction. [Sørensen 2012] showed that the problem of finding a single proper equilibrium in a bimatrix game can be done by applying Lemke's algorithm to a linear complementarity problem of polynomial size, and hence is PPAD-complete. The discussion in sections 5 and 6 of this paper provide an answer to the question of how small ϵ needs to be to guarantee that an *ϵ-proper* equilibrium profile is indeed close to a proper equilibrium, though this bound in general cannot be determined efficiently. This was briefly mentioned as an issue in [Sørensen 2012], although the method proposed in that paper did not require this information. [Belhaiza et al. 2012] propose ad-hoc methods for computing proper equilibria by first solving a mixed 0-1 quadratic program and analytically verifying solution candidates. They note that the informal analytical approach of iteratively satisfying the criteria of *ϵ-properness* was introduced by [Myerson 1991]. This paper provides a formal generalization of this method using the language of hyperplane arrangements as it applies to the problem of verifying fine tenability. This characterization is, to the best of my knowledge, not fleshed out in any of the previous literature, and sheds light on the geometry of fine tenability and proper equilibria. This characterization allows for a clean discussion of the complexity of verifying fine tenability, and also provides explicit bounds for the $\bar{\epsilon}$ used in the definition of fine tenability (See definition 3 below) in bimatrix games.

2 PRELIMINARIES

I consider finite games of complete information defined as $G = (N, S, u)$. The set N denotes the set of players, with $|N| = n$. The set $S = \times_{i=1}^{N} S_i$ denotes the set of pure strategy profiles, with S_i denoting the set of pure strategies for player i. The mapping $u_i : S \rightarrow \mathbb{R}$ assigns to every pure strategy profile s a payoff for player i. The space of mixed strategies for player i is given by ΔS_i, the collection of probability distributions over strategies in S_i, with generic elements given by σ_i. The set of mixed strategy profiles is denoted $\Sigma = \times_{i=1}^{N} \Delta S_i$, with generic element σ. For a mixed strategy profile, the payoff to player i is given by,

$$u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s)$$

where $\sigma(s) = \Pi_{i=1}^{N} \sigma_i(s_i)$. With the usual abuse of notation, $u_i(s_i, s_{-i})$ denotes the payoff of player i from playing $s_i \in S_i$ when all other players play according to $s \in S$. Similarly, $u_i(s_i, \sigma_{-i})$ is defined as the payoff to player i when she plays pure strategy s_i while all other players randomize according to $\sigma \in \Sigma$.

For a given mixed strategy σ_i, let $Y_i(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$ denote the *carrier* of σ_i. For a mixed strategy profile σ, let $BR_i(\sigma) = \{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s_i^{'}, \sigma_{-i}) \ \ \forall s_i^{'} \in S_i\}$ denote the set of best responses for player i to the profile σ. A profile σ is a *Nash Equilibrium* of $G = (N, S, u)$ if and only if for each player i, $Y_i(\sigma_i) \subseteq BR_i(\sigma)$.

In this paper, I discuss the robust strategy blocks introduced in [Myerson and Weibull 2015]. In particular, I am interested in the verification of blocks that are said to be *coarsely tenable* and *finely tenable*. I make no nondegeneracy assumptions about the finite game in question. Before proceeding to the relevant equilibrium concepts, I introduce the framework of *consideration-set games* from [Myerson and Weibull 2015].

For a set E, let $\mathbb{P}(E)$ denote the power set of E. Let C_i denote a generic element of $\mathbb{P}(S_i)$. Every set C_i is referred to as a "type" for player i. If a player is of type C_i, then strategies within C_i are precisely the strategies that player i considers using. All strategies outside of C_i are played with probability 0 if player i is of type C_i. There is a probability distribution μ_i over types for each player. That is, $\mu_i(C_i)$ denotes the probability that player i is of type C_i. These distributions are independent for each player. The joint distribution over $\mathbb{P}(S)$ is denoted by μ. This defines a consideration set game $G^\mu = (N, S, u, \mu)$.

A pure strategy for player i is a mapping $f_i : \mathbb{P}(S_i) \to S_i$ with the property that $f_i(C_i) \in C_i$. Denote the space of all pure strategies for player i by F_i. Mixed strategies are defined in the usual way, as randomizations over the pure strategy mappings. The associated space of mixed strategies for player i is denoted by ΔF_i. A generic element of ΔF_i is denoted by τ_i, and a strategy profile of the consideration set game is denoted by τ. Given τ, let $\tau_{i|C_i}$ denote the conditional probability distribution over pure strategies in C_i when player i is of type C_i. Each profile τ induces a probability distribution τ^μ over the pure strategies of each player in the following way

$$\tau^\mu(s_i) = \sum_{\{C_i : s_i \in C_i\}} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

This projection of τ onto the space of mixed strategies allows one to compute the payoffs to each player given a profile τ. For a profile τ, denote the induced mixed strategy profile by τ^μ.

A profile τ is said to be an equilibrium of the consideration-set game if for each player i and each $C_i \in \mathbb{P}(S_i)$,

$$u_i(\tau_{i|C_i}, \tau^\mu_{-i}) = max_{s_i \in C_i} u_i(s_i, \tau^\mu_{-i})$$

That is to say, if player i is of type C_i then player i chooses only best responses (from among those strategies contained in C_i) to the probability distribution induced by τ_{-i}. The equilibria are defined ex-ante. The following definitions will refer to subsets (referred to as "blocks") $T = \times_{i=1}^N T_i$ of the collection of pure strategy profiles. Each $T_i \subseteq S_i$ is a nonempty collection of pure strategies for each player.

Definition 1 ([Myerson and Weibull 2015]). A strategy block $T \subseteq S$ is said to be *coarsely tenable* if $\exists\ \bar{\epsilon} \in (0, 1)$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and each type distribution μ such that $\mu(T) > 1 - \epsilon$,

$$max_{s_i \in S_i} u_i(s_i, \tau^\mu_{-i}) = max_{t_i \in T_i} u_i(t_i, \tau^\mu_{-i})$$

for player i and each equilibrium τ of the incomplete information game induced by μ.

Coarse tenability requires that the block T be robust against all sufficiently small deviations from the convention T. The notion of *fine tenability* is a weaker concept in that it requires robustness against a specific class of deviations from convention. In particular, the deviations that are said to be $\epsilon - proper$. The definition of ϵ-properness for strategy blocks is quite similar to the one defined for strategy profiles defined by Myerson in his definition of *proper* equilibria (see [Myerson 1978]). Indeed, the definition imposes a requirement on the relative probability of trembles between "more" and "less" rational alternatives of strategy blocks. Here, rationality is used to describe the number of strategic alternatives that a player considers when playing as an unconventional type.

Definition 2. [Myerson and Weibull 2015] Given a strategy block T and an $\epsilon \in (0,1)$, a type distribution μ is said to be *ϵ-proper* if, for every player i:

- $\mu_i(T_i) > 1 - \epsilon$
- $\mu_i(C_i) > 0, \quad \forall C_i \subseteq S_i$
- $T_i \neq C_i \subset D_i \implies \mu_i(C_i) \leq \mu_i(D_i)\epsilon$

Where C_i and D_i are arbitrary subsets of S_i.

The first point in the definition requires that players are sufficiently likely to be of a conventional type. The second imposes a requirement that every possible type has a positive probability under μ. The third point reflects the requirement that players place greater probability on more rational deviations from convention.

I now introduce the final set-valued equilibrium concept of fine tenability.

Definition 3. [Myerson and Weibull 2015] A block T is *finely tenable* if $\exists\ \bar{\epsilon} \in (0,1)$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and for any type distribution μ that is $\epsilon - proper$,

$$max_{t_i \in T_i} u_i(t_i, \tau^{\mu}_{-i}) = max_{s_i \in S_i} u_i(s_i, \tau^{\mu}_{-i})$$

for each player i and each equilibrium τ of the of the incomplete information game induced by μ.

As noted by Myerson and Weibull (2015), if a block T is coarsely tenable, then it also satisfies the weaker condition of fine tenability. It is important to note that as $\mu(T) \longrightarrow 1$, the projections of equilibria of the consideration-set games G^{μ} converge to equilibria of the support restricted game $G^T = (N, T, u)$. This fact is given in [Myerson and Weibull 2015], and is used throughout the analysis in this paper.

3 CHARACTERIZATION USING PERTURBATIONS

In order to obtain a more concrete understanding of the structure of these block concepts and relate them to other solution concepts in the game theory literature, it is helpful to rephrase them in terms of the more familiar language of strategically perturbed games. A perturbation vector $\delta_i \in \mathbb{R}_+^{|S_i|}$ assigns to every strategy $s_i \in S_i$ a minimum probability $\delta_i(s_i)$ that this pure strategy must receive at any mixed strategy profile. These perturbation vectors must of course satisfy $\sum_{s_i \in S_i} \delta_i(s_i) \leq 1$. Given a tuple of perturbations $\{\delta_i\}_{i=1}^{N}$, the set of admissible mixed strategies for player i is defined as $\{\sigma_i \in \Delta S_i : \sigma_i(s_i) \geq \delta_i(s_i) \ \ \forall s_i \in S_i\}$. Let $Y_i^{\delta}(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > \delta_i(s_i)\}$ denote the *δ-carrier* of the mixed strategy σ.

Definition 4. An equilibrium of the perturbed game $G = (N, S, u, \delta)$ is a mixed strategy profile σ such that for each player i, $Y_i^{\delta}(\sigma_i) \subseteq BR_i(\sigma)$.

With the appropriate definitions established, I proceed with relating projections of equilibria of consideration-set games to Nash equilibria of appropriately perturbed games. Keeping in mind that the final goal is to examine the properties of tenable strategy sets, I will suppose throughout this section that the analysis is given with an arbitrary, but fixed, strategy block T.

Lemma 1. Fix a block T, an arbitrary distribution over types μ, and an equilibrium of the associated consideration-set game τ. Consider the induced probability distribution over pure strategies τ^{μ}. *Claim:* There is a probability distribution μ' such that the marginal probability distributions μ'_i have the property that

- $\mu'_i(T_i) = \mu_i(T_i)$

- $\mu_i'(C_i) = 0$ for all C_i such that $C_i \neq T_i$ and $|C_i| \neq 1$

(μ_i' places positive probability only on T_i and individual strategies in S_i) and there is an equilibrium τ' of the consideration-set game induced by μ' with $\tau^\mu = \tau'^\mu$. That is to say, τ' induces the same probability distribution over pure strategies as τ.

All proofs are in the appendix. The idea is that for a given equilibrium τ with projection τ^μ, reconstruct the distribution τ^μ directly using type distributions over individual strategies and T_i. Distributions of the form μ' described in the above lemma will be referred to as *simple* distributions throughout the rest of this paper. Note that when discussing *simple* distributions over types one requires the context of a specific T in order for the definition to make sense. The correct terminology would be "*simple with respect to T*." I omit this in future mentions of *simple* distributions with the understanding that the distributions are *simple* with respect to the block T in question.

Since deviations of the form described in the above lemma must be considered anyway, Lemma 1 says that there is no loss in generality in restricting attention to such simplified probability distributions. These distributions can be viewed in some sense as the familiar strategy perturbations used in describing various equilibrium concepts in the literature (*proper*, *(strictly) perfect*, etc.). The precise connection will soon become clear. For a mixed strategy profile σ, let $BR_i^T(\sigma) = \{s_i \in T_i : u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \ \ \forall s_i' \in T_i\}$ denote the set of best responses among strategies in T_i for player i against profile σ.

Definition 5. For a fixed strategy block T and perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$, a profile σ is a *T-equilibrium* of the perturbed game (N, S, u, δ) if for each player i, $Y_i^\delta(\sigma_i) \subseteq BR_i^T(\sigma)$.

Note that *T-equilibria* need not coincide with Nash equilibria of the perturbed game. However, these are precisely the profiles which correspond to projections of equilibria of consideration-set games as $\mu(T) \longrightarrow 1$ in a strong sense.

Lemma 2. For a fixed block T in the game $G = (N, S, u)$ and for every *simple* type distribution μ with $\mu_(T) > 0$, there is a perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$ with $\sum_{s_i \in S_i} \delta_i(s_i) = 1 - \mu_i(T_i)$ for each player i and such that for every equilibrium τ of the consideration-set game G^μ, there is a *T-equilibrium* σ of the perturbed game (N, S, u, δ) such that $\tau^\mu = \sigma$.

A variant of the reverse direction of Lemma 2 is also true.

Lemma 3. For a fixed strategy block T, each perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$ and every *T-equilibrium* σ of the perturbed game $G = (N, S, u, \delta)$, there is a *simple* type distribution μ such that $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ and such that there is an equilibrium τ of the consideration-set game G^μ in which $\sigma = \tau^\mu$

Lemmas 2 and 3 provide a simple way to express the relationship between consideration-set games and strategically perturbed games. This correspondence allows for the comparison of the set-valued equilibrium concepts found in [Myerson and Weibull 2015] to other solution concepts found in the game theory literature. In particular, I show that every *coarsely tenable* strategy block T contains a set that is *stable* in the sense of [Kohlberg and Mertens 1986] (henceforth *KM-stable*). Before proceeding, I provide a definition of *KM-stability* used in [Jansen and Vermeulen 2001] for the verification of KM-stable sets in bimatrix games.

Definition 6 (Jansen and Vermeulen (2001))**.** A closed set E of strategy pairs is called a *KM-set* if for each neighborhood V of E, there is $\eta > 0$ such that if the strategy perturbations satisfy $||\delta|| < \eta$, then there is an equilibrium of the perturbed game lying in V. A set E that is minimal with respect to this property is referred to as *(KM-)stable.* [1]

Note here that $||\delta||$ refers to the Euclidean norm of the full tuple of perturbations $\{\delta_i\}_{i=1}^N$ in $\times_{i=1}^N \mathbb{R}^{|S_i|}$.

Proposition 1. In a finite game $G = (N, S, u)$, if a strategy block T is *coarsely tenable*, then T contains a set that is *KM-stable.*

It is worth nothing that this result is known to be true for generic finite games (see [Wikman 2016]). I will return to the relationship between consideration-set games and perturbed games when discussing the verification of coarse tenability in the next section. I now move on to characterizing the relationship between projections of equilibria of consideration-set games with ϵ*-proper* type distributions and tuples (σ, δ) with specific structure.

Definition 7. Consider a fixed strategy block T and $\epsilon > 0$. A tuple (σ, δ) consisting of a mixed strategy profile σ and a strategy perturbation δ is said to be $T(\epsilon)$*-proper* if the following conditions hold:

- $s_i \in argmax_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \geq \delta_i(s_i)$
- $s_i \notin argmax_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) = \delta_i(s_i)$
- $\delta_i(s_i) > 0$
- $u_i(s_i, \sigma_{-i}) > u_i(s_j, \sigma_{-i}) \Rightarrow \delta_i(s_i)\epsilon \geq \delta_i(s_j)$
- $\sum_{\{s_i \in S_i\}} \delta_i(s_i) \leq \epsilon$

Note that the profile σ need not be $\epsilon - proper$, however the perturbations δ satisfy a properness criterion under the ordering over strategies induced by σ. The idea behind much of the work in the rest of this section is once again to consider all probability that arises from deviations from convention T as perturbations. The proof that for a fixed T the equilibria of consideration-set games with $\epsilon - proper$ type distributions admit tuples which are $T(\epsilon) - proper$ is almost exactly the same proof as Proposition 2 of [Myerson and Weibull 2015]. A proof is provided here for purposes of completeness and to note some subtle differences.

Lemma 4. Fix a strategy block T. Suppose that μ is an ϵ^3*-proper* type distribution and that τ is an equilibrium of the consideration-set game G^μ. Then there is a tuple (σ, δ) that is $T(\epsilon)$*-proper* and such that $\tau^\mu = \sigma$.

Now, I would like to show that any tuple that is $T(\epsilon)$*-proper* corresponds to an equilibrium of a consideration-set game with an ϵ*-proper* type distribution. Unfortunately, this in general requires type distributions which are proper according to $\epsilon^{\frac{1}{z}} > \epsilon$ for a given $T(\epsilon)$*-proper* tuple (σ, δ) and $z \in \mathbb{N}$. I will need to introduce some notation before continuing. Consider a mixed strategy profile σ. This profile induces a total preorder over the strategies of each player according to expected payoffs. Let $\succsim_i$ denote the total preorder of the form $s_i \succsim_i s_j$ if and only if $u_i(s_i, \sigma_{-i}) \geq u_i(s_j, \sigma_{-i})$ with $\sim$ denoting indifference. This total preorder partitions the strategy space of player i into indifference classes. Denote these indifference classes by $\zeta_1^i, \zeta_2^i, ..., \zeta_{r_i}^i$ with $r_i \leq |S_i|$. The classes are ordered such that $s \in \zeta_k^i$ and $s' \in \zeta_{k+1}^i$ implies $u_i(s, \sigma_{-i}) < u_i(s', \sigma_{-i})$.

Proposition 2. For a fixed strategy block T, there exists $\epsilon^* > 0$ and $z \in \mathbb{N}$ such that for all $\epsilon \in (0, \epsilon^*)$ and any tuple (σ, δ) that is $T(\epsilon)$*-proper*, there is an $\epsilon^{\frac{1}{z}}$*-proper* type distribution and an equilibrium τ of the associated consideration-set game such that $\tau^\mu = \sigma$.

[1] Note that Kohlberg and Mertens (1986) required that the set E be composed only of Nash equilibria. Imposing minimality on E ensures that the sets considered in Jansen and Vermeulen (2001) are indeed the sets defined as *stable* under the definition of Kohlberg and Mertens.

Remark 1. The proof of this proposition is essentially an application of the following observation. Fix $k \in \mathbb{N}$ and $M > 0$. There is an $\bar{\epsilon} \in (0,1)$ and $z \in \mathbb{N}$ such that for any collection $\alpha_0, \alpha_1, ..., \alpha_k \in [0, M]$ and $\epsilon \in (0, \bar{\epsilon})$,

$$\sum_{i=0}^{k} \frac{\alpha_i}{\epsilon^{\frac{1}{z}}} \leq \frac{1}{\epsilon}$$

Note that precisely how small ϵ^* in Proposition 2 needs to be is unimportant. Since fine tenability only considers behavior under small perturbations, it suffices to show that there is an equivalence between the two concepts in a neighborhood of the game where T is played with probability 1.

4 VERIFICATION OF COARSELY TENABLE BLOCKS

Given a strategy block T, a natural question is how to determine whether it is *coarsely tenable*. This section is devoted to determining how to answer this question in the case of two players. Throughout the remainder of this paper, I consider a bimatrix game in which player 1 has pure strategy set S_1 and player 2 has pure strategy set S_2. I suppose that $|S_1| = m$ and $|S_2| = n$. The associated mixed strategy sets are given by ΔS_1 and ΔS_2. The set of mixed strategy profiles is denoted by $\Sigma = \Delta S_1 \times \Delta S_2$. A generic element of ΔS_1 is denoted by p, while a generic element of ΔS_2 is denoted by q. Given a profile (p, q), payoffs are determined by the matrices A_{mxn} and B_{mxn} in the following way:

$$u_1(p, q) = pAq$$

$$u_2(p, q) = pBq$$

Nash equilibria of this game are the strategy pairs (p, q) such that

$$pAq \geq p'Aq \;\; \forall p' \in \Delta S^1$$

$$pBq \geq pBq' \;\; \forall q' \in \Delta S^2$$

Throughout this section, I assume that I am armed with a concise description of the Nash equilibria of the full bimatrix game. The set of all Nash equilibria in a bimatrix game is a finite union of convex polytopes using results of [Jansen 1981]. Furthermore, there exist methods of computing all Nash equilibria of a bimatrix game in finite time, regardless of assumptions on nondegeneracy (see [Avis et al. 2010]).

I now move on to the discussion of verifying whether a given block T satisfies the requirements imposed by *coarse tenability*. Much of this discussion is based on the work of [Jansen and Vermeulen 2001] who described a method of verifying whether a set is *stable* under the definition of [Kohlberg and Mertens 1986].

Consider a block T with the property that all equilibria of the block game G^T are also equilibria of the entire bimatrix game G. I am interested in determining whether it is coarsely tenable. This entails determining how the equilibria with support contained in the block T change as the distribution over types μ places small probability on blocks other than T. Lemma 1 implies that is suffices to consider deviations from convention T in which players consider individual pure strategies rather than larger strategy blocks. Lemmas 2 and 3 show that this problem can, with some care, be phrased as a problem about profiles in strategically perturbed games. Using the methodology of [Jansen and Vermeulen 2001], sub-blocks of T can be used to determine precisely how the equilibria of the perturbed game change, and thus how the equilibria of the consideration set game change as a result of altering the distribution over type. In what follows in Lemma 5 as well as throughout the rest of this paper, I refers to an indexed set of strategies in S_1. I sometimes use $i \in I$ to refer to a strategy $s_i \in I$ within this indexed set. I let $p_i = \sigma_i(s_i)$ for a strategy $s_i \in S_i$ and e_i denotes a basis vector corresponding to playing strategy s_i with probability 1. Similar notation is used for an indexed set $J \subseteq S_2$ and strategies for player 2.

Lemma 5. Consider a fixed strategy block T. A profile (p, q) is a *T-equilibrium* for the perturbed (bimatrix game) $G = (N, S, u, \delta)$ if and only if there is a nonempty sub-block $I \times J$ with $I \subseteq T_1$ and $J \subseteq T_2$ such that $(p, \delta_1) \in S^1_{IJ}$ and $(q, \delta_2) \in S^2_{IJ}$, where

$$S^1_{IJ} = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T_2 \\ p_i \geq \delta_1(s_i) & \forall i \in I \\ p_i = \delta_1(s_i) & \forall i \notin I \\ 0 \geq -\delta_1(s_i) & \forall i \\ \sum_i p_i = 1 & \end{cases}$$

$$S^2_{IJ} = \begin{cases} e_iAq - e_kAq \geq 0 & i \in I, k \in T_1 \\ q_j \geq \delta_2(s_j) & \forall j \in J \\ q_j = \delta_2(s_j) & \forall j \notin J \\ 0 \geq -\delta_2(s_j) & \forall j \\ \sum_j q_j = 1 & \end{cases}$$

Since Lemmas 1-3 established a correspondence between *T-equilibria* and projections of equilibria of consideration-set games, every solution (p, q) to the above system given δ yields an equilibrium of a consideration-set game in which $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ for each player i. For the case in which $|T_i| \geq 2$ for each player, the solutions yield equilibria in which $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$.

For a fixed sub-block $I \times J$, S^1_{IJ} describes a convex polytope in $\mathbb{R}^{2m}$, and S^2_{IJ} describes a convex polytope in $\mathbb{R}^{2n}$. To determine whether a given strategy block T is coarsely tenable, it suffices to enumerate only those sub-block $I \times J$ which possess a solution at $\delta = 0$. Indeed, the fact that equilibria of the incomplete information game induced by a type distribution μ converge to equilibria of the block game G^T as $\mu(T) \longrightarrow 1$ implies that these are the only sub-blocks that are relevant to the problem. Any pair $I \times J$ which does not possess a solution tuple at $\delta = 0$ is such that there is no equilibrium of the block game in which each of the strategies in I and J are best responses. Then at least one of the associated polytopes is bounded away from the zero perturbation (full information) region. Since coarse tenability imposes restrictions only on perturbations sufficiently close to 0, such pairs $I \times J$ can safely be ignored.

Given a concise description of the equilibria of the full game and that T is such that each equilibria of the block game G^T are also equilibria of the full game, it is not difficult to enumerate the pairs $I \times J \subseteq T$ for which there are solutions at $\delta = 0$. Fix such a pair $I \times J$. In order to verify whether coarse tenability is violated, it suffices to determine whether there is a sequence of solutions $\{(p_k, q_k, \delta_k)\}_{k=1}^{\infty}$ with $\delta_k \longrightarrow 0$ and such that for each (p_k, q_k), there is some player i and strategy $s_k \in S_i \backslash T_i$ such that s_k does strictly better than all strategies in T_i. The Bolzano-Weierstrass theorem implies that there exists a sequence in which the same strategy $s_k = s^*$ plays this role. Such a strategy, if it exists, must then be a (weak) best reply to a solution of the system when $\delta = 0$. Let us suppose for the moment that such a strategy s^* exists and without loss of generality that it belongs to player 2, and denote its index in S_2 by ω. Fix some strategy $s_j \in J$. Evidently the above sequence must lie entirely within the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$. [2] Using the fact that the hyperplane $\{(p, \delta) : pBe_\omega - pBe_j = 0\}$ intersects S^1_{IJ} at $\delta = 0$ and the fact that a sequence of

[2]Note that implicitly this uses the fact that each player must be indifferent between all elements of I and J at any solution of the system.

solutions in which s^* does strictly better than s_j exists, it must be that S^1_{IJ} possesses a vertex that lies in the half space $pBe_\omega - pBe_j > 0$. In fact, this is the only instance in which such a sequence may exist. If all vertices lie in the half space $pBe_\omega - pBe_j \le 0$, then the hyperplane $pBe_\omega - pBe_j = 0$ is either tangent to S^1_{IJ} at $\delta = 0$, or aligns with one of its faces. In either case, s^* does not constitute a violation of coarse tenability.

Proposition 3. Fix a strategy block T with the property that all equilibria of the block game G^T are equilibria of the entire game G. Enumerate all sub-blocks $I \times J$ of T for which S^1_{IJ} and S^2_{IJ} possess solutions when $\delta = 0$. For each such sub-block and for each player, enumerate all strategies outside of T that are (weak) best responses to an equilibrium with support contained within $I \times J$, denote the set of such strategies as $\{\Omega^1_{IJ}, \Omega^2_{IJ}\}$.
Claim: T is coarsely tenable if and only if for each such sub-block $I \times J$:

(1) For each strategy $s_\omega \in \Omega^2_{IJ}$, S^1_{IJ} does not possess a vertex that lies in the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$, where ω denotes the index of s_ω in S_2 and j denotes the index of an arbitrary strategy in J.
(2) For each strategy $s_\omega \in \Omega^1_{IJ}$,$S^2_{IJ}$ does not possess a vertex that lies in the half-space $\{(q, \eta) : e_\omega Aq - e_i Aq > 0\}$, where ω denotes the index of s_ω in S_1 and i denotes the index of an arbitrary strategy in I.

This characterization of coarse tenability resembles the *undominated* characterization of perfect equilibria in bimatrix games due to van Damme (see Theorem 3.2.2 of [van Damme 1983]).

5 VERIFICATION OF FINELY TENABLE BLOCKS

Recall the definition of fine tenability introduced in Section 2 before continuing. In order to determine whether a given block T is finely tenable, it often suffices to show that it is coarsely tenable. Indeed, [Myerson and Weibull 2015] show that the concepts of coarse and fine tenability agree for generic normal-form games. While this is a useful result, much of the usefulness of fine tenability as a solution concept stems from its bite in normal form representations of extensive form games. Furthermore, determining whether or not a game is of a given generic form, for instance the class of *nondegenerate* games or the class of *hyper-regular* games introduced in [Myerson and Weibull 2015], can be computationally expensive. With this in mind, a discussion on fine tenability seems appropriate. Fortunately, Proposition 2 has already supplied a useful characterization for checking the conditions of fine tenability. In order to verify whether a given block T is finely tenable, it is helpful to return to the sets S^1_{IJ} and S^2_{IJ}. It once again suffices to consider only those sub-blocks $I \times J$ of T that possess solutions when perturbations are set to zero. [3]

Fix such a sub-block $I \times J$. I am interested in determining whether there exists a sequence $\epsilon_k \longrightarrow 0$ and a sequence of $T(\epsilon_k) - proper$ tuples (σ_k, δ_k) which solve the systems S^1_{IJ} and S^2_{IJ} and are such that there is some player i and $s_\omega \in S_i \backslash T_i$ with $u_i(s_\omega, \sigma_{-i,k}) > max_{t_i \in T_i} u_i(t_i, \sigma_{-i,k})$ for all k. Such a sequence $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ will be called an $I \times J$*-refutation.* If no such sub-block possesses a refutation, then T is finely tenable.

Proposition 4. A strategy block T is finely tenable if and only if every sub-block $I \times J$ of T does not possess an $I \times J$*-refutation.*

Proposition 4 implies that the problem of verifying whether a block T is finely tenable reduces to the problem of finding $I \times J$*-refutations* among particular sub-blocks of T. However, this does not provide any indication of how to determine whether such refutations exist for a fixed T. Evidently,

[3]Note that while coarse tenability imposes the restriction that block equilibria are also equilibria of the entire game, fine tenability does not require this. See for instance Example 2 of [Myerson and Weibull 2015]

if an $I \times J$ refutation exists, then it is possible to construct a refutation with the desirable property that the total preorders $\gtrsim_{i,k}$ induced by σ_k are constant for all k. This suggests that a geometric approach to the problem of searching for refutations could be fruitful.

Fix a sub-block $I \times J$ of the strategy block T. For a given total preorder $\gtrsim_2$ for player 2, the set of strategies of player 1 that induce this preorder is given by $p(\gtrsim_2)$. Note that in general $p(\gtrsim_2)$ is not closed, however the closure of this set is a polytope. Under the terminology of [Jansen 1993], the strategies in $p(\gtrsim_2)$ are said to be *order equivalent*. Indeed, the space of mixed strategies for each player can be partitioned into equivalence classes using *order equivalence* as an equivalence relation.

Now, suppose there is an $I \times J$-*refutation* $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ with constant total preorders $\gtrsim_1, \gtrsim_2$. Then $\sigma_{1,k} \in p(\gtrsim_2)$ and $\sigma_{2,k} \in q(\gtrsim_1)$ for all k. Furthermore, $\delta_{1,k}$ is $\epsilon_k - proper$ according to $\gtrsim_1$ for all k. Then it must be that $S^1_{IJ} \cap p(\gtrsim_2)$ and $S^2_{IJ} \cap q(\gtrsim_1)$ are nonempty and contain the perturbations $\{\delta_{1,k}\}_{k=1}^{\infty}$ and $\{\delta_{2,k}\}_{k=1}^{\infty}$ respectively.[4] The question is then: given a set $S^1_{IJ} \cap p(\gtrsim_2)$, according to what total preorders $\gtrsim_1$ does this set contain $\epsilon - proper$ perturbations δ_1 for all $\epsilon > 0$? Indeed, an answer to this question would provide a brute force method to the problem of verifying fine tenability. Namely, check all possible pairings of total preorders and determine whether the corresponding perturbations exist within S^1_{IJ} and S^2_{IJ} for all $\epsilon > 0$. The following results characterize necessary and sufficient conditions on $S^1_{IJ} \cap p(\gtrsim_2)$ to guarantee that it supports $\epsilon - proper$ perturbations of a given total preorder $\gtrsim_1$ for all $\epsilon > 0$. In what follows, the expression $v(\delta_1(s))$ denotes the value of $\delta_1(s)$ at a vertex v of a polytope.

Lemma 6. Fix a sub-block $I \times J$ of a block T, $\gtrsim_1$ and $\gtrsim_2$. The set $S^1_{IJ} \cap p(\gtrsim_2)$ contains $T(\epsilon) - proper$ tuples (σ_1, δ_1) according to the order $\gtrsim_1$ for all $\epsilon > 0$ only if the following conditions hold

- There is a vertex of $cl(S^1_{IJ} \cap p(\gtrsim_2))$ that lies on the zero perturbation region (i.e. $v(\delta_1(s)) = 0$ for all $s \in S_1$).
- For each strategy $s \in S_1$, there is a vertex of $cl(S^1_{IJ} \cap p(\gtrsim_2))$ in which $v(\delta_1(s')) = 0$ and $v(\delta_1(s)) > 0$ for all $s' \prec_1 s$.

Before proving sufficiency, I will require the following lemma.

Lemma 7. For a fixed sub-block $I \times J$ of a block T and total preorders $\{\gtrsim_1, \gtrsim_2\}$,

$$relint(cl(S^1_{IJ} \cap p(\gtrsim_2))) \subseteq S^1_{IJ} \cap p(\gtrsim_2)$$

where *relint*$(\cdot)$ denotes the relative interior of a set (under the usual topology).

Lemma 8. The conditions of Lemma 6 are sufficient to guarantee that for $\epsilon > 0$, there exists a tuple (σ_1, δ_1) that is $T(\epsilon) - proper$ according to $\gtrsim_1$ and that lies in the relative interior of $cl(S^1_{IJ} \cap p(\gtrsim_2))$.

With these results about the geometry of $T(\epsilon) - proper$ tuples, I am able to provide an exact method for determining the presence of $I \times J$-*refutations* for any sub-block of T. Before describing the algorithm, I provide some basic definitions of the relevant objects used in the procedure.

Definition 8. The hyperplane arrangements, in $\mathbb{R}^{2|S^1|}$ and $\mathbb{R}^{2|S^2|}$ respectively, associated with the block $I \times J$ are defined as

$$H^1_{IJ} = \left\{ \sum_i p_i = 1 \right\} \cup \left\{ \bigcup_{\{i,j \in S^2, i<j\}} \{pBe_i - pBe_j = 0\} \right\} \cup \left\{ \bigcup_{i \in S^1} \{\delta_i = 0\} \cup \{p_i = \delta_i\} \right\}$$

[4] Here, $p(\gtrsim_2)$ and $q(\gtrsim_1)$ are taken to be those elements (p, δ_1) and (q, δ_2) in which p induces $\gtrsim_2$ over S_2 and q induces $\gtrsim_1$ over S_1.

$$H^2_{IJ} = \left\{ \sum_j p_j = 1 \right\} \cup \left\{ \bigcup_{\{i,j \in S^1, i<j\}} \{e_i A q - e_j A q = 0\} \right\} \cup \left\{ \bigcup_{j \in S^2} \{\eta_j = 0\} \cup \{q_j = \eta_j\} \right\}$$

The vertices of these hyperplane arrangements will be useful when computing vertices of the polytopes used in the procedure. Since it is unknown whether vertex enumeration can be done in polynomial time, it seems sensible to compute the vertices of this arrangement once and use them to quickly compute the vertices of one of the (possibly exponentially many) polytopes that are used in the procedure, rather than perform vertex enumeration at each step. Indeed, given a total preorder $\succsim_2$, the vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$ will be a subset of the vertices of H^1_{IJ}.

Definition 9. The zero perturbation regions of S^1_{IJ} and S^2_{IJ} projected onto the space of mixed strategies for each player are given by

$$Z^1_{IJ} = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T^2 \\ p_i \geq 0 & \forall i \in I \\ p_i = 0 & \forall i \in S^1 \backslash I \\ \sum_i p_i = 1 & \end{cases}$$

$$Z^2_{IJ} = \begin{cases} e_i A q - e_k A q \geq 0 & i \in I, k \in T^1 \\ q_j \geq 0 & \forall j \in J \\ q_j = 0 & \forall j \in S^2 \backslash J \\ \sum_j q_j = 1 & \end{cases}$$

These sets are used to prune the number of total preorders used in the procedure. In particular, it suffices to only focus on those total preorders which are locally possible given Z^1_{IJ} and Z^2_{IJ}. For instance, if Z^1_{IJ} is a singleton then any strict preferences in the total preorder $\succsim_2$ induced at this point will be preserved as $\epsilon \longrightarrow 0$. The algorithm is outlined on the next page.

Remark 2. It is important to note what is meant by "associated inequalities" in step $6(a)$ below. The associated inequalities are those found by relaxing all strict inequalities imposed by the total preorders into weak inequalities. In order to ensure that this would in fact result in the appropriate closures, it suffices to check that $S^1_{IJ} \cap p(\succsim_2)$ and $S^2_{IJ} \cap q(\succsim_1)$ are nonempty. This can of course be done by checking any convex combination of the vertices found in step $6(a)$ in which all vertices receive strictly positive weight.

Proposition 5. Algorithm 1 correctly determines in finite time whether the sub-block $I \times J$ of T possesses a refutation.

A number of comments on the method are in order. It is in principle possible to verify whether a pair of total preorders $\{\succsim_1, \succsim_2\}$ constitute a refutation by solving a pair of linear programs (see lemma 13). On the other hand, enumerating all total preorder pairs and performing this procedure very quickly becomes infeasible. The first 5 steps of the procedure will in general (at least in nondegenerate games) drastically reduce the number of pairs to check. It is of course possible to refine and improve upon this pruning of the total preorders by using the structure of the face lattice of the associated hyperplane arrangements alongside the restrictions imposed by properness. These improvements deserve some attention, and are a subject of ongoing research.

Algorithm 1 here is in some sense a formal generalization of the informal method of analytically determining whether a given equilibrium is proper given in [Myerson 1991] and fleshed out a bit more in [Belhaiza et al. 2012]. The method here is adapted to a different solution concept, but the logic behind lemmas 6-8 and the procedure defined above are applicable to the problem of verifying proper equilibria. The formalization given here does provide a number of insights into the geometry of the problem, however.

The algorithm described here is exponential in the worst case, requiring the comparison of an exponential number of total preorder pairings. The next result shows that it is NP-hard to determine whether a block possesses a refutation, even when it consists of single strategies. The following reduction modifies the one provided by [Hansen and Lund 2018], who in turn modify a reduction given by [Conitzer and Sandholm 2008]. I construct a game in which a given block possesses a refutation if and only if an instance of a Boolean satisfiability problem in conjunctive normal form with exactly three literals per clause possesses no satisfying assignment.

Algorithm 1: Search for IJ-refutations

(1) Compute the vertices V_H^i of H_{IJ}^i and V_Z^i of Z_{IJ}^i using a vertex enumeration algorithm for hyperplane arrangements, e.g. [Avis and Fukuda 1992]
(2) Set $j_1 \sim_2 j_2 \;\; \forall j_1, j_2 \in J$ and $i_1 \sim_1 i_2 \;\; \forall i_1, i_2 \in I$
(3) For each $k_1, k_2 \in S_2 \backslash J, k_1 \neq k_2$:
  (a) If $V_Z^1 \subset pBe_{k_1} - pBe_{k_2} > 0$, set $k_1 \succ_2 k_2$
  (b) Else if $V_Z^1 \subset pBe_{k_1} - pBe_{k_2} < 0$, set $k_2 \succ_2 k_1$
  (c) Else place no restriction on k_1 and k_2 regarding $\succsim_2$
(4) Print restrictions on $\succsim_2$
(5) Repeat steps $3 - 4$ for restrictions on $\succsim_1$
(6) For each pair of total preorders $\succsim_1$ and $\succsim_2$ satisfying the restrictions from steps $2 - 5$ and such that there is some player i and strategy $s_\omega \in S_1(S_2)$ with $s_\omega \succ_i s$ for some $s \in I(J)$
  (a) Enumerate the vertices of $cl(S_{IJ}^1 \cap p(\succsim_2))$ and $cl(S_{IJ}^2 \cap q(\succsim_1))$ by checking which vertices of H_{IJ}^1 and H_{IJ}^2 satisfy the associated inequalities.
  (b) Check whether the vertices of $cl(S_{IJ}^1 \cap p(\succsim_2))$ satisfy the conditions of Lemma 6 with respect to $\succsim_1$ to determine existence of $T(\epsilon) - proper$ perturbations for ϵ near zero. Repeat for $cl(S_{IJ}^2 \cap q(\succsim_1))$ with respect to $\succsim_2$.
  (c) If the requisite conditions are satisfied for both $cl(S_{IJ}^1 \cap p(\succsim_2))$ and $cl(S_{IJ}^2 \cap q(\succsim_1))$ then a refutation has been found, and the procedure halts after printing the pairing of total preorders $\{\succsim_1, \succsim_2\}$.
  (d) Else there is no refutation for the total preorder pairing
(7) If no pairing of total preorders yields a refutation, then the block $I \times J$ possesses no refutations.

Definition 10. Let ϕ be a Boolean formula in conjunctive normal form with exactly three literals per clause. Let V denote the set of variables, with $|V| = n$. Let L denote the set of literals, with $+l_i$ and $-l_i$ denoting the positive and negative literals corresponding to variable x_i. Let the mapping $v : L \longrightarrow V$ be defined by $v(+l_i) = v(-l_i) = x_i$, assigning to any literal its corresponding variable. Denote the set of clauses by C. Finally, each player will have three additional pure strategies denoted by f, g, and h. The symmetric, normal form game $G(\phi)$ that I consider is defined, with payoffs to the row player, in Table 1 below. The block to be considered is $T = \{g\} \times \{g\}$. In what follows, I will show that T is finely tenable if and only if the formula ϕ does not have a satisfying assignment.

	l'	x'	h'	c'	f'	g'
l	$n-1$ if $l \neq -l'$ $n-4$ if $l = -l'$	$n-1$ if $v(l) \neq v$ $n-4$ if $v(l) = v$	$n-1$	$n-1$	$n-1$	$n-3$
x	$n-1$ if $v(l) \neq v$ n if $v(l) = v$	$n-1$	$n-1$	$n-1$	$n-1$	$n-2$
h	$n-1-\frac{1}{2n}$	$\frac{(n-1)^2+(n-4)}{n}$	$n-1$	$n-1$	$n-1$	$n-3$
c	1 if $l \notin c$ 0 if $l \in c$	$n-1$	n	$n-1$	$n-1$	$n-1$
f	$\frac{n-\frac{1}{2}}{n}$	$n-1$	$n-1$	$n-1$	$n-1$	$n-1$
g	$n-1$	$n-1$	$n-1$	n	0	$n-1$

Table 1. The game $G(\phi)$

Lemma 9. Let (σ_1, σ_2) denote any $T(\epsilon) - proper$ profile for ϵ small. Then for each player i and each pair of variables x_j and x_k,

$$\sigma_i(+l_j) + \sigma_i(-l_j) = \sigma_i(+l_k) + \sigma_i(-l_k)$$

and

$$\sigma_i(x_j) = \sigma_i(x_k)$$

Lemma 10. If (σ_1, σ_2) is a $T(\epsilon) - proper$ profile with ϵ small, then there is a partition of the set of literals L into two sets L_1 and L_2 such that for each player

$$\sigma_i(l_1) = \sigma_i(l_1') \ \ \forall\, l_1, l'_1 \in L_1$$
$$\sigma_i(l_2) = \sigma_i(l_2') \ \ \forall\, l_2, l'_2 \in L_2$$
$$l_j \in L_1 \ \Rightarrow\ -l_j \in L_2$$
$$l_1 \gtrsim_i l_2 \ \ \forall l_1 \in L_1,\ l_2 \in L_2$$

Lemma 11. If ϕ has a satisfying assignment then $T = \{g\} \times \{g\}$ is not finely tenable.

Lemma 12. If $T = \{g\} \times \{g\}$ is finely tenable, then ϕ does not have a satisfying assignment.

Proposition 6. It is NP-hard to verify whether a given sub-block $I \times J$ of a block T possesses an *$I \times J$-refutation.*

Proposition 7. It is NP-hard to verify whether a given block T is finely tenable.

Note that proposition 4 alongside lemmas 6-8 show that one can in fact view *$I \times J$-refutations* as a pair of total preorders over the pure strategies of each player. If one is provided with a pair of total preorders, one for each player, it is not difficult to determine whether it serves as an $I \times J$ refutation. This provides a proof for NP-completeness of the problem of verifying fine tenability.

Lemma 13. Given a block T, a sub-block $I \times J$, and a pair of total preorders $\gtrsim_1$, $\gtrsim_2$, one can determine whether $\{\gtrsim_1, \gtrsim_2\}$ serves as an $I \times J$ refutation for T by solving a polynomial number of linear programs, each of which contains only a polynomial number of contraints.

Proposition 8. The problem of verifying whether a given block T is finely tenable in a bimatrix game is NP-complete.

6 APPLICATION TO PROPER EQUILIBRIA

The results of Lemmas 6-8, used here to analyze fine tenability, can also be applied to proper equilibria. As I will soon show, it gives rise to an exact method of verifying whether a given equilibrium constitutes a proper equilibrium of a bimatrix game. Before proceeding, I introduce some basic definitions for the section.

Definition 11. [Myerson 1978] A mixed strategy profile σ is said to be an *ϵ-proper equilibrium* if for each player i the following conditions hold:

- $\sigma_i(s) > 0 \ \ \forall s \in S_i$
- $u_i(s, \sigma_{-i}) > u_i(s', \sigma_{-i}) \Rightarrow \sigma_i(s)\epsilon \geq \sigma_i(s') \ \ \forall s, s' \in S_i$

A Nash equilibrium σ is said to be a *proper equilibrium* if there exists a sequence $\{\epsilon_k, \sigma^k\}_{k=1}^{\infty}$ such that

- $\epsilon_k > 0$ and $\lim_{k\to\infty} \epsilon_k = 0$
- σ^k is an ϵ_k*-proper equilibrium*
- $\lim_{k\to\infty} \sigma^k = \sigma$

The procedure below determines whether a given input Nash equilibrium $\sigma = (p^*, q^*)$ is a proper equilibrium. It is based on the logic of Lemmas 6-8 above that it is possible to determine whether a convergent sequence of *ϵ-proper* equilibria exists by analyzing the faces of a pair of hyperplane arrangements. Define the hyperplane arrangements:

$$\tilde{H}^1 = \left\{ \sum_i p_i = 1 \right\} \cup \left\{ \bigcup_{\{j,k \in S^2 : j<k\}} \{pBe_j - pBe_k = 0\} \right\} \cup \left\{ \bigcup_i \{p_i = 0\} \right\}$$

$$\tilde{H}^2 = \left\{ \sum_j q_j = 1 \right\} \cup \left\{ \bigcup_{\{i,k \in S^1 : i<k\}} \{e_iAq - e_kAq = 0\} \right\} \cup \left\{ \bigcup_j \{q_j = 0\} \right\}$$

Consider the following sets:

$$P^1(p^*, q^*) = \begin{cases} pBe_j - pBe_k \geq 0 & j \in Y(q^*), k \in S_2 \\ p_i \geq 0 & \forall i \in S^1 \\ \sum_i p_i = 1 \end{cases}$$

$$P^2(p^*, q^*) = \begin{cases} e_iAq - e_kAq \geq 0 & i \in Y(p^*), k \in S_1 \\ q_j \geq 0 & \forall j \in S^2 \\ \sum_j q_j = 1 \end{cases}$$

where $Y(\cdot)$ denotes the carrier of a mixed strategy. Each closed face (of positive dimension) f_1 of $\tilde{H}^1$ such that $p^* \in f_1$ and $f_1 \subseteq P^1(p^*, q^*)$ is associated with a total preorder $\succsim_2$, the preference ordering of player 2 over strategies in S^2 induced by mixed strategies that lie in the relative interior of f_1. A similar statement holds for faces f_2 of $\tilde{H}^2$. To determine whether (p^*, q^*) is a proper equilibrium, it suffices to check all pairings of such faces f_1 and f_2 for a set of conditions similar to those given in Lemma 6. In what follows, let $\succsim_{i,f_{-i}}$ denote the total preorder over strategies for player i induced by strategies on the relative interior of f_{-i}. For a given total preorder $\succsim_{i,f_{-i}}$, let ζ_k^i denote the k^{th} indifference class, with ζ_1^i denoting the best strategies for player i. Let r_i denote the total number of indifference classes in S_i according to the total preorder $\succsim_i$.

Lemma 14. A Nash equilibrium $\sigma = (p^*, q^*)$ is a *proper* equilibrium if and only if there is a pair of faces f_1 (of $\tilde{H}^1$) and f_2 (of $\tilde{H}^2$) such that

- $p^* \in f_1$ and $q^* \in f_2$
- $f_1 \subseteq P^1(p^*, q^*)$ and $f_2 \subseteq P^2(p^*, q^*)$
- For each player i, each indifference class ζ^i_k and each strategy $s \in \zeta^i_k$, there is a vertex of f_i in which $p_s > 0$ and $p'_s = 0$ for all $s' \in \zeta^i_l$ with $l > k$.

7 DISCUSSION

The characterization in Section 3 provides a straightforward method of verifying coarse tenability of a given block T regardless of degeneracy assumptions on the game. A natural method of finding coarsely tenable blocks is then to determine all strategy blocks of the game which possess the property that all block equilibria are also equilibria of the full game, and then determining whether the block satisfies the additional conditions imposed by coarse tenability. Enumerating such blocks for nondegenerate games is straightforward, albeit computationally expensive. The relationship between projections of equilibria of consideration-set games and the usual notion of strategic perturbations has allowed me to relate coarse tenability to KM-stable sets. It would be interesting to explore the relationship between tenable strategy sets and other familiar solution concepts in the game theory literature (see Wikman (2016) for work in this area).

The methods developed in Section 4 are based on simple geometric arguments about *proper* strategy profiles. Note that the procedures for verifying the conditions of fine tenability and proper equilibria, while exact, may require calculations involving extremely small numbers, and bounds for these values can be articulated using the geometry of bimatrix games; although, it be may computationally expensive to do so. This is the main drawback of the methods proposed here. On the other hand, it is quite useful to have bounds on the level of precision required for verifying these concepts. For instance, the results of this paper can be applied to the mixed 0-1 quadratic programming problem proposed in [Belhaiza et al. 2012] for the verification of proper equilibria.

There are a number of avenues in which the current paper can be improved. First, it would be very interesting to have a procedure which computes minimal strategy sets that satisfy the criteria of the two solution concepts discussed in this paper, even for bimatrix games. In particular, a procedure that is motivated by a population of agents collectively steering toward a stable convention seems the most promising. Note that, unlike sets which are closed under rational belief (CURB), the intersection of two coarsely (finely) tenable blocks need not be coarsely (finely) tenable in bimatrix games. This appears to preclude a procedure along the lines of [Benisch et al. 2010] in which a minimal block can be constructed by iteratively including additional strategies, as now the final product may not minimal under set-inclusion (this is due to the fact that the ordering of the addition of new strategies to the block is now important).

It is worth noting that complexity results for coarse tenability are also available, and NP-hardness of the problem of verifying coarse tenability can be proved by a straightforward modification of the reduction given in [Conitzer and Sandholm 2008]. A formal procedure for verifying coarse tenability is also a natural next step in the analysis. The procedure given here for fine tenability can also be improved upon a great deal. As mentioned in section 5, it is possible to use the properties of the face semi-lattice of the associated hyperplane arrangement to reduce the number of total preorders enumerated. The exact nature of this improvement is a subject of ongoing research.

REFERENCES

David Avis and Komei Fukuda. 1992. A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra. *Discrete and Computational Geometry* 8, 3 (Sept. 1992), 295–313.

Davis Avis, Gabriel Rosenberg, Rahul Savani, and Bernhard von Stengel. 2010. Enumeration of Nash Equilibria for Two-Player Games. *Economic Theory* 42, 1 (Jan. 2010), 9–37.

Kaushik Basu and Jörgen Weibull. 1991. Strategy Subsets Closed Under Rational Behavior. *Economics Letters* 36, 2 (June 1991), 141–146.

Slim Belhaiza, Charles Audet, and Pierre Hansen. 2012. On Proper Refinement of Nash Equilibria for Bimatrix Games. *Automatica* 48, 2 (Feb. 2012), 297–303.

Michael Benisch, George David, and Tuomas Sandholm. 2010. Algorithms for Closed Under Rational Behavior (CURB) Sets. *Journal of Artificial Intelligence Research* 38 (Aug. 2010), 513–534.

Dimitri Bertsekas, Angelia Nedic, and Asuman Ozdaglar. 2003. *Convex Analysis and Optimization.* Athena Scientific, Belmont, MA.

Felix Brandt and Markus Brill. 2016. Computing Dominance-Based Solution Concepts. *ACM Transactions on Economics and Computation* 5, 2 (Oct. 2016).

Xi Chen and Xiaotie Deng. 2006. Settling the Complexity of Two-Player Nash Equilibrium. In *Proceedings of the Annual Symposium on Foundations of Computer Science (FOCS '06).* ACM, Berkeley, CA.

Vincent Conitzer and Tuomas Sandholm. 2008. New Complexity Results About Nash Equilibria. *Games and Economic Behavior* 63, 2 (July 2008), 621–641.

Constantinos Daskalakis, Paul Goldberg, and Christos Papadimitrou. 2006. The Complexity of Computing a Nash Equilibrium. In *Proceedings of the Annual Symposium on Theory of Computing (STOC '06).* ACM, Seattle, WA.

John Duggan and Michel Le Breton. 2014. *Choice-Theoretic Solutions for Strategic Form Games.* Retrieved February 10, 2020 from https://sites.google.com/view/johnduggan/other-papers?authuser=0

Itzhak Gilboa and Eitan Zemel. 1989. Nash and Correlated Equilibria: Some Complexity Considerations. *Games and Economic Behavior* 1, 1 (March 1989), 80–93.

Kristoffer Hansen. 2017. The Real Computational Complexity of Minmax Value and Equilibrium Refinements in Multi-player games. In *Algorithmic Game Theory (LNCS) (SAGT '17).* Springer, Cham.

Kristoffer Hansen and Troels Lund. 2018. Computational Complexity of Proper Equilibrium. In *Proceedings of the 2018 ACM Conference on Economics and Computation (EC '18).* ACM, New York, NY.

Kristoffer Hansen, Peter Miltersen, and Troels Sørensen. 2010. The Computational Complexity of Trembling Hand Perfection and Other Equilibrium Refinements. In *Proceedings of the third international conference on Algorithmic Game Theory (SAGT '10).* ACM, Athens, Greece.

Mathijs Jansen. 1981. Maximal Nash Subsets for Bimatrix Games. *Naval Research Logistics Quarterly* 28, 1 (March 1981).

Mathijs Jansen. 1993. On the Set of Proper Equilibria of a Bimatrix Game. *International Journal of Game Theory* 22, 2 (June 1993), 97–106.

Mathijs Jansen and Dries Vermeulen. 2001. On the Computation of Stable Sets and Strictly Perfect Equilibria. *Economic Theory* 17, 2 (March 2001), 325–344.

Max Klimm, Tuomas Sandholm, and Jörgen Weibull. 2011. *Computational Aspects of Closure Under Rational Behavior in Games.* https://sites.google.com/site/joergenweibull/working-papers

Elon Kohlberg and Jean-Francois Mertens. 1986. On the Strategic Stability of Equilibria. *Econometrica* 54, 5 (Sept. 1986), 1003–1037.

Peter Bro Miltersen and Troels Bjerre Sørensen. 2010. Computing a quasi-perfect equilibrium of a two-player game. *Economic Theory* 42, 1 (2010), 175–192.

Roger Myerson. 1978. Refinements of the Nash Equilibrium Concept. *International Journal of Game Theory* 7, 2 (June 1978), 73–80.

Roger Myerson. 1991. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, MA.

Roger Myerson and Jörgen Weibull. 2015. Tenable Strategy Blocks and Settled Equilibria. *Econometrica* 83, 3 (May 2015), 943–976.

John Nash. 1951. Non-Cooperative Games. *Annals of Mathematics* 54, 2 (Sept. 1951), 286–295.

Troels Sørensen. 2012. Computing a Proper Equilibrium of a Bimatrix Game. In *In ACM Conference on Electronic Commerce (EC '12).* ACM, Valencia Spain.

Eric van Damme. 1983. *Refinements of the Nash Equilibrium Concept.* Springer, New York, New York.

John von Neumann and Oskar Morgenstern. 1944. *Theory of Games and Economic Behavior.* Princeton University Press, Princeton, New Jersey.

Peter Wikman. 2016. *Nash Blocks.* https://sites.google.com/wikman.com/peter/research?authuser=0

A APPENDIX OF PROOFS

Proof of Lemma 1

Proof. Suppose that $|T_i| \geq 2$ for each player i. Construct μ' using the projection τ^μ as follows :

$$\mu_i'(T_i) = \mu_i(T_i) \;\; \forall i$$

$$\mu_i'(s_i) = \sum_{\{C_i : s_i \in C_i, C_i \neq T_i\}} \tau_{i|C_i}(s_i)\mu_i(C_i) \;\; \forall s_i \in S_i \text{ and } \forall i$$

$$|C_i| \neq 1 \;\wedge\; C_i \neq T_i \;\Rightarrow \mu_i'(C_i) = 0$$

Now τ' has essentially been constructed for us. Given the restrictions imposed by μ', τ_i' is pinned down for all sets other than T_i for each player i. Finally, set $\tau'_{i|T_i} = \tau_{i|T_i}$. To see that the proof is complete, note that by construction $\tau^\mu = \tau'^\mu$, the equilibria induce the same distribution over pure strategies for each player. Therefore since $\tau_{i|T_i}$ was part of an equilibrium, it must also be that $\tau'_{i|T_i}$ is as well.

For the case in which $T_i = s_i$ for some player i, define $\mu_i^{'}(T_i) = \tau^\mu(s_i)$. □

Proof of Lemma 2

Proof. Fix a *simple* type distribution μ and suppose for now that $|T_i| \geq 2$ for each player i. Let τ be an equilibrium of the consideration-set game G^μ with associated projection τ^μ. Define $\delta_i(s_i) = \mu_i(s_i)$ for each strategy $s_i \in S_i$ and each player i. Define $\sigma_i(s_i) = \tau^\mu(s_i)$. To see that σ is a *T-equilibrium*, note that if $s_i \notin BR_i^T(\sigma)$ then the fact that τ is an equilibrium implies that $\tau_{i|T_i}(s_i) = 0$ and hence $\sigma_i(s_i) = \mu_i(s_i) = \delta_i(s_i)$ which implies that $s_i \notin Y_i^\delta(\sigma_i)$.

For the case in which $|T_i| = 1$ for some player i, define $\delta_i(s_i) = 0$ for such players. □

Proof of Lemma 3

Proof. Suppose that $|T_i| \geq 2$ for each player i. Given δ, set $\mu_i(s_i) = \delta_i(s_i)$ for each $s_i \in S_i$ and each player i. Set $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ for each player i. Define

$$\tau_{i|T_i}(s_i) = \frac{\sigma_i(s_i) - \delta_i(s_i)}{\mu_i(T_i)} \;\; \forall s_i \in T_i$$

for each player i. The fact that μ is assumed to be a *simple* type distribution implies that τ is pinned down for all blocks aside from T. Thus by construction

$$\tau_i^\mu(s_i) = \sigma_i(s_i)$$

and furthermore the fact that σ is a *T-equilibrium* implies that all strategies $s_i \in T_i$ such that $\tau_{i|T_i}(s_i) > 0$ are an element of $BR_i^T(\tau^\mu)$.

For the case in which $|T_i| = 1$ for some player i, set $\mu_i(T_i) = 1 - \sum_{s_i \notin T_i} \delta_i(s_i)$. Note that τ_i is automatically constructed for these players since μ is *simple*. □

Proof of Proposition 1

Proof. I will show that T contains a KM-set, and hence must contain a minimal KM-set. Let E denote the set of equilibria of the block game G^T (which are also equilibria of the entire game under the assumption of coarse tenability). Since E is the entire set of equilibria of a block game, it is closed.

Suppose, seeking contradiction, that this set E is not a KM-set. Then there is a neighborhood V of E in the space of mixed strategies such that for any $\eta > 0$, there is a perturbation δ with $||\delta|| < \eta$ and such that all equilibria of the game perturbed under δ lie outside of V. Thus there exists a sequence of perturbations $\{\delta_k\}_{k=1}^{\infty}$ such that $||\delta_k|| > 0$ for each k and $||\delta_k|| \to 0$, and such that for each k, the perturbed game game (N, S, u, δ) possesses no equilibria within V.

Suppose that $|T_i| \geq 2$ for each player i. Interpret the perturbations $\{\delta_{i,k}\}_{i=1}^{N}$ as a *simple* type distribution μ_k with respect to T for a moment, with $\mu_{i,k}(T_i) = 1 - \sum_{s_i \in S_i} \delta_{i,k}(s_i)$ and $\mu_{i,k}(s_i) = \delta_{i,k}(s_i)$. Recall that the projections of equilibria of the associated consideration-set games converge to equilibria of the block game G^T as $\mu(T) \to 1$ (i.e. $k \to \infty$). Hence there is a $K_1 > 0$ such that for all $k \geq K_1$, there is an equilibrium of the consideration-set game induced by the *simple* distribution μ_k that lies in the neighborhood of V. Now, if I can show that the projections of these equilibria of the consideration-set game induced by μ_k are (eventually) also equilibria of the perturbed game (N, S, u, δ), the proof will be complete. This is precisely where the assumption of coarse tenability has bite. Indeed, there is $\bar{\epsilon} > 0$ such that if $\mu(T) > 1 - \bar{\epsilon}$, then all projections of equilibria of the consideration-set game are also equilibria of the associated perturbed game. To see why, recall that the profile σ is an equilibrium of the perturbed game (N, S, u, δ) if and only if σ satisfies the minimum probability constraints imposed by δ and for each player i:

$$Y_i^{\delta}(\sigma_i) \subseteq BR_i(\sigma)$$

Let $K_2 > 0$ be such that for all $k \geq K_2$, $\mu_k(T) > 1 - \bar{\epsilon}$, and let $K^* = max\{K_1, K_2\}$. Consider any equilibrium τ of a consideration-set game μ_k for $k \geq K^*$. Then

$$\tau_{i|T_i}(s_i) > 0 \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

since T is coarsely tenable. This then implies that

$$\tau_i^{\mu_k}(s_i) > \mu_k(s_i) = \delta_{i,k}(s_i) \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

Thus $\tau^{\mu_k} = \sigma$ constitutes an equilibrium of the perturbed game (N, S, u, δ) that lies within the set V, a contradiction.

For the case in which $|T_i| = 1$ for some player i, define $\mu_{i,k}(T_i) = 1 - \sum_{s_i \notin T_i} \delta_{i,k}(s_i)$ for such players. The rest of the argument is the same. □

Proof of Lemma 4

Proof. Let $|T_i| \geq 2$ for each player. For a player i and given strategy $s_i \in S_i$, let $\beta_i(s_i) \subseteq \mathbb{P}(S_i)$ denote the collection of sets $C_i \in \mathbb{P}(S_i)$ such that

$$s_i \in argmax_{t_i \in C_i} u_i(t_i, \tau_{-i}^{\mu})$$

It is important to distinguish between two cases:

- Case 1: There exist two strategies $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^{\mu}) < u_i(t_i, \tau_{-i}^{\mu})$ and such that there is $C_i^* \in \beta_i(r_i)$ with $T_i = C_i^* \cup \{t_i\}$.
  In this case, note that t_i is the unique best response in T_i. Define

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

for all $s_i \neq t_i$. Define

$$\delta_i(t_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(t_i)\mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*)$$

Define $\sigma_i = \tau_i^{\mu}$. Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i)\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq C_i^*\}} \mu_i(C_i \cup \{t_i\}) + \mu_i(C_i^*) \leq \epsilon\delta_i(t_i)$$

Now let s_i be such that $u_i(s_i, \tau_{-i}^{\mu}) > u_i(t_i, \tau_{-i}^{\mu})$. Then

$$\delta_i(t_i) \leq \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i \cup \{s_i\}) + \epsilon\mu_i(T_i \cup s_i) \leq \epsilon\delta_i(s_i)$$

Verifying all other pairings uses the same logic, and is more simple due to the fact that T_i does not appear in the calculations. See case 2 for how this is done explicitly. Note that

$$\sum_{s_i \in S_i} \delta_i(s_i) = \sum_{C_i \neq T_i} \mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*) \leq \epsilon^3 + \epsilon^2\mu_i(T_i) \leq \epsilon$$

- Case 2: There do not exist two strategies $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^{\mu}) < u_i(t_i, \tau_{-i}^{\mu})$ and such that there is $C_i^* \in \beta_i(r_i)$ with $T_i = C_i^* \cup \{t_i\}$.
  In this case, it suffices to set

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

  Then let $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^{\mu}) < u_i(t_i, \tau_{-i}^{\mu})$. Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i \cup \{t_i\}) \leq \epsilon\delta_i(t_i)$$

  The case in which $|T_i| = 1$ for some player i only requires slight modifications, and only in the case in which the strategy $s_i = T_i$ is among the *worst* strategies in S_i. In such a case, choose $\delta_i(s_i)$ to be a sufficiently small positive number, and construct δ_i the same way as above for all other strategies.

□

Proof of Proposition 2

Proof. Suppose without loss of generality that $m = |S_1| = max_i\{|S_i|\}$. Fix $\epsilon^* = \frac{1}{m^2 2^{2m}}$ and $z = 2m$. Let (σ, δ) be a $T(\epsilon) - proper$ tuple for $\epsilon < \epsilon^*$. I will construct a type distribution μ that is $\epsilon^{\frac{1}{z}} - proper$ as well as an equilibrium τ of the associated consideration-set game directly. Enumerate the indifference classes $\zeta_1^i, \zeta_2^i, ..., \zeta_{r_i}^i$ induced by σ for each player i. For a given strategy s, let $\zeta(s)$ denote the indifference class that strategy s is an element of. The construction below is for player 1, without any loss of generality.

I will distinguish between two cases.

- Case 1: $T_1 \neq \bigcup_{l \leq k} \zeta_l^1$ for any k.
  Define the term:

$$R(s) = \sum_{\{C \subset \cup_{\{s' \precsim s\}} s',\ C \neq T_1\ :\ s \in C\}} \frac{1}{\epsilon^{\frac{|C|-1}{z}} |C \cap \zeta(s)|}$$

Note that for strategies in the same indifference class, R can differ only if one is an element of T and the other is not. If $|\zeta_1^1| = 1$, let s_1 denote its element and define $\mu_1(s) = \delta_1(s) := \gamma_1^1$ for all $s \in S^1$. Otherwise let

$$\mu_1(s) = min\left[\frac{min_{\{s\in\zeta_1^1\}}\delta_1(s)}{max_{\{s\in\zeta_1^1\}}R(s)}, \frac{\sum_{s\in\zeta_1^1}\delta_1(s)}{\sum_{s\in\zeta_1^1}R(s)+\frac{1}{\epsilon^{\frac{|\zeta_1^1|-1}{z}}}}\right] := \gamma_1^1$$

for all $s \in S^1$. Set $\mu_1(C) = \frac{\mu_1(s)}{\epsilon^{\frac{|C|-1}{z}}}$ for all C such that $|C| < |\zeta_1^1|$ and $C \neq T_1$. Define

$$\mu_1(C) = \sum_{s\in\zeta_1^1}\delta_1(s) - \gamma_1^1\sum_{s\in\zeta_1^1}R(s) := \gamma_2^1$$

for all $C \neq T_1$ such that $|C| = |\zeta_1^1|$. For all subsets C such that $|\zeta_1^1| < |C| < |\zeta_1^1| + |\zeta_2^1|$, set $\mu_1(C) = \frac{\gamma_2^1}{\epsilon^{\frac{|C|-|\zeta_1^1|}{z}}}$. For any strategy s, define

$$\bar{R}(s) = \sum_{\{C\subset \cup_{\{s'\lesssim s\}}s',\ C\neq T_1\ :\ s\in C\}}\frac{\mu_1(C)}{|C\cap\zeta(s)|}$$

Here, $\bar{R}(s)$ can be interpreted as the probability that a strategy s receives from all blocks in which it is among the best alternatives, aside from the largest such block. This is of course under the assumption that the conditional distribution over the best alternatives in a block is uniform. When moving to the largest block in which a given strategy is a best alternative, the conditional distribution will no longer be uniform. This will be shown in the construction of τ_1 later on.

Let

$$\mu_1(C) = \sum_{s\in\zeta_2^1}\delta_1(s) - \sum_{s\in\zeta_2^1}\bar{R}(s) := \gamma_3^1$$

for all sets $C \neq T_1$ such that $|C| = |\zeta_1^1|+|\zeta_2^1|$. Now it is possible to give an iterative construction of μ_1 for all subsets. For all sets $C \neq T_1$ such that $\sum_{l=1}^{k}|\zeta_l^1| < C < \sum_{l=1}^{k+1}|\zeta_l^1|$ for some $k < r_1$, define

$$\mu_1(C) = \frac{\gamma_{k+1}^1}{\epsilon^{\frac{|C|-\sum_{l=1}^{k}|\zeta_l^1|}{z}}}$$

where

$$\gamma_k^1 = \sum_{s\in\zeta_{k-1}^1}\delta_1(s) - \sum_{s\in\zeta_{k-1}^1}\bar{R}(s),\quad 2\leq k\leq r_1+1$$

and $\mu_1(C) = \gamma_{k+1}^1$ if $C \neq T_1$ and $|C| = \sum_{l=1}^{k}|\zeta_l^1|$. This completes μ_1.

Now we must construct τ_1. Fix a strategy $s \in S^1$. For each set $C \neq T_1$ such that $s \in C$, $s \gtrsim s'$ for all $s' \in C$ and $C \neq \bigcup_{s'\lesssim s} s'$, define

$$\tau_{1|C}(s) = \frac{1}{|\zeta(s)\cap C|}$$

For the set $C = \bigcup_{s' \precsim s} s'$, define

$$\tau_{1|C}(s) = \frac{\delta_1(s) - \bar{R}(s)}{\mu_1(C)}$$

Finally, set

$$\tau_{1|T}(s) = \frac{\sigma_1(s) - \delta_1(s)}{\mu_1(T)}$$

where $\mu_1(T) = 1 - \sum_{C \neq T_1} \mu_1(C)$. Note that in order for this construction to work, it must be that

$$\bar{R}(s) \leq \delta_1(s)$$

for each strategy s. Suppose $s \in \zeta_1^1$, then by the definition of γ_1^1 one has that

$$\bar{R}(s) = \gamma_1^1 R(s) \leq \frac{\delta_1(s)}{R(s)} R(s) = \delta_1(s)$$

If $s \in \zeta_k^1$ for $k \geq 2$, then

$$\bar{R}(s) = \sum_{\{C \subset \cup_{\{s' \precsim s\}} s',\ C \neq T_1\ :\ s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta(s)|} \leq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}} 2^{|S^1|} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}} 2^{|S_1|}$$

Using the fact that for all $s \in \zeta_k^1$ and $s' \in \zeta_{k-1}^1$,

$$\delta_1(s)\epsilon \geq \delta_1(s')$$

it must be that

$$\bar{R}(s) \leq \frac{m\delta_1(s)\epsilon}{\epsilon^{\frac{m}{z}}} 2^{|S_1|} = m\delta_1(s) 2^{|S_1|} \epsilon^{\frac{z-m}{z}} = m\delta_1(s) 2^{|S_1|} \epsilon^{\frac{1}{2}} \leq m\delta_1(s) 2^{|S_1|} \epsilon^{*\frac{1}{2}} = \frac{m\delta_1(s) 2^{|S_1|}}{m 2^m} \leq \delta_1(s)$$

There are still some details to verify. In particular, to show that $\tau_1^{\mu_1} = \sigma_1$, and that μ_1 is $\epsilon^{\frac{1}{z}} - proper$. For a strategy s, its probability under this specification is given by

$$\tau_1^{\mu_1}(s) = \bar{R}(s) + \delta_1(s) - \bar{R}(s) + \sigma_1(s) - \delta_1(s) = \sigma_1(s)$$

To verify properness, it suffices to show that for all $k \geq 2$

$$\gamma_{k+1}^1 \epsilon^{\frac{1}{z}} \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}}$$

or

$$\gamma_{k+1}^1 \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}}$$

One has that

$$\gamma_{k+1}^1 = \sum_{s \in \zeta_k^1} (\delta_1(s) - \bar{R}(s)) \geq \sum_{s \in \zeta_k^1} \delta_1(s) - \frac{2^{|S^1|} - 1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \sum_{s' \in \zeta_{k-1}^1} \delta_1(s')$$

Furthermore

$$\frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}}$$

Finally,

$$\sum_{s\in\zeta_k^1}\delta_1(s) \geq \frac{max_{s'\in\zeta_{k-1}^1}\delta_1(s')}{\epsilon^{\frac{z-|\zeta_k^1|}{z}}\epsilon^{\frac{|\zeta_k^1|}{z}}} \geq \frac{max_{s'\in\zeta_{k-1}^1}\delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}}\frac{1}{\epsilon^{*\frac{1}{2}}} \geq \frac{max_{s'\in\zeta_{k-1}^1}\delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}}m2^{|S_1|} \geq \frac{\sum_{s'\in\zeta_{k-1}^1}\delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}}2^{|S_1|}$$

as desired. As a final note, γ_1^1 was constructed so that properness holds for $k=1$.

- Case 2: $T=\bigcup_{l\leq k}\zeta_l^1$ for some k. It will be helpful to identify $k_{T_1}=max\{k : T_1\cap\zeta_k^1\neq\varnothing\}$. For all $k<k_{T_1}-1$, the construction is identical to Case 1. If $|\zeta_{k_{T_1}}^1|\geq 2$, then $k_{T_1}-1$ can also be constructed in the same way as Case 1. To handle instances in which $|\zeta_{k_{T_1}}^1|\geq 2$, one needs only to modify the construction from Case 1 for the sets

$$\mathbf{C}_{k_{T_1}}=\{C : C\subset\bigcup_{l\leq k_{T_1}}\zeta_l^1,\ |C|=-1+\sum_{l\leq k_{T_1}}|\zeta_k^1|\}$$

This modification is due to the fact that it is no longer possible to use the set

$$\bigcup_{l\leq k_{T_1}}\zeta_k^1=T$$

to manage residual probabilities. Instead, these remaining probabilities must be distributed among the blocks in $\mathbf{C}_{k_{T_1}}$. To formalize this idea, define a modified version of $\bar{R}$ for strategies in $\zeta_{T_1}^1$. For $s\in\zeta_{k_{T_1}}^1$, let

$$\bar{R}_{k_{T_1}}(s)=\sum_{\{C\subset\cup_{s'\lesssim s}s',\ |C|\leq -2+\sum_{l\leq k_{T_1}}|\zeta_l^1|,\ s\in C\}}\frac{\mu_1(C)}{|C\cap\zeta_{k_{T_1}}^1|}$$

where $\mu_1(C)$ is constructed in the same way as Case 1 above for all sets $|C|\leq -2+\sum_{l\leq k_{T_1}}|\zeta_l^1|$ by using the γ_l^1. Now partition $\mathbf{C}_{k_{T_1}}$ into two separate subsets.

$$\mathbf{C}_{k_{T_1}}^{*}=\{C : C\in\mathbf{C}_{k_{T_1}},\ \zeta_{k_{T_1}}^1\subseteq C\}$$

$$\mathbf{C}_{k_{T_1}}^{**}=\{C : C\in\mathbf{C}_{k_{T_1}},\ \zeta_{k_{T_1}}^1\nsubseteq C\}$$

Note that each element of $\mathbf{C}_{k_{T_1}}^{**}$ contains all strategies in $\zeta_{k_{T_1}}^1$ except for one. The idea is to use the smallest possible probability, while maintaining properness, for elements in $\mathbf{C}_{k_{T_1}}^{*}$. Then use $\mathbf{C}_{k_{T_1}}^{**}$ to sort the rest of the remaining probabilities out. In particular, let

$$\mu_1(C)=\frac{\gamma_{k_{T_1}}^1}{\epsilon^{\frac{|\zeta_{k_{T_1}}^1|-1}{z}}}$$

$$\tau_{1|C}(s)=\frac{1}{|\zeta_{k_{T_1}}^1|}$$

for all $C\in\mathbf{C}_{k_{T_1}}^{*}$ and $s\in\zeta_{k_{T_1}}^1$. For elements of $\mathbf{C}_{k_{T_1}}^{**}$, there is always one strategy from $\zeta_{k_{T_1}}^1$ that is excluded. Enumerate the elements of $\zeta_{k_{T_1}}^1$ as $s_1,s_2,...,s_{|\zeta_{k_{T_1}}^1|}$. Given C, suppose that $s_i\notin C$, then define

$$\mu_1(C) = \delta_1(s_{i+1}) - \bar{R}_{k_{T_1}}(s_{i+1}) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}} |\zeta^1_{k_{T_1}}|} \sum_{l \leq k_{T_1}-1} |\zeta^1_l|$$

$$\tau_{1|C}(s_{i+1}) = 1$$

where with an abuse of notation $s_{|\zeta^1_{k_{T_1}}|+1} = s_1$. Now to verify that

$$\mu_1(C) \geq \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}$$

for all $C \in \mathbf{C}^{**}_{k_{T_1}}$ in order to maintain properness. This is of course verified in the same way as Case 1. For any $s \in \zeta^1_{k_{T_1}}$,

$$\delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}} |\zeta^1_{k_{T_1}}|} \sum_{l \leq k_{T_1}-1} |\zeta^1_l| \geq \delta_1(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}(2^{|S_1|} - 1)$$

Thus I would like to say that

$$\delta_1(s) \geq \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}} 2^{|S_1|}$$

To conclude,

$$\frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}} 2^{|S_1|} \leq \frac{\sum_{s' \in \zeta^1_{k_{T_1}-1}} \delta_1(s')}{\epsilon^{\frac{m}{z}}} 2^{|S_1|} \leq \frac{m\delta_1(s)\epsilon^*}{\epsilon^{*\frac{1}{2}}} 2^{|S_1|} = \delta_1(s)$$

Thus the distribution up to this point is $\epsilon^{\frac{1}{z}} - proper$. Define

$$\gamma^1_{k_{T_1}+1} = max_{s \in \zeta^1_{k_{T_1}}} \left[\delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}} |\zeta^1_{k_{T_1}}|} \sum_{l \leq k_{T_1}-1} |\zeta^1_l| \right] \frac{1}{\epsilon^{\frac{1}{z}}}$$

With $\gamma^1_{k_{T_1}+1}$ defined, the rest of the construction is the same as Case 1, with the special note that

$$\mu_1(C) = \gamma^1_{k_{T_1}+1} \epsilon^{\frac{1}{z}}$$

for all C such that $C \notin \mathbf{C}_{k_{T_1}}$ with $|C| = |T| - 1$.

The sub-case in which $|\zeta^1_{k_{T_1}}| = 1$ requires only slight modifications to the above argument.

It is immediate that $\tau^{\mu_i}(s) = \sigma_i(s)$ for each player i and each strategy $s \in S^i$. Furthermore, since conditional distributions $\tau_{i|C}$ are such that only best responses in C receive positive probability, the profile τ constructed above is indeed an equilibrium of the consideration-set game.

□

Proof of Lemma 5

Proof. To show necessity, suppose that (p, q) is a *T-equilibrium* under perturbations $\{\delta_1, \delta_2\}$. Let $I = Y_1^\delta(p) \subseteq T_1$ and $J = Y_2^\delta(q) \subseteq T_2$. Note that for all strategies outside of I (J), it must be that $p_i = \delta_1(s_i)$ ($q_j = \delta_2(s_j)$). Finally note that $I \subseteq BR_1^T(p, q)$ and $J \subseteq BR_2^T(p, q)$. Hence $(p, \delta_1) \in S_{IJ}^1$ and $(q, \delta_2) \in S_{IJ}^2$.

To show sufficiency, suppose that $(p, \delta_1) \in S_{IJ}^1$ and $(q, \delta_2) \in S_{IJ}^2$ for some strategy sub-block $I \times J$ of T. Clearly then $Y_i^\delta(p, q) \subseteq BR_i^T(p, q)$ for each player i, and hence the tuple (p, q) is a *T-equilibrium* since the other constraints imposed by S_{IJ}^1 and S_{IJ}^2 ensure that these mixed strategies lie in the set of admissible strategies given the perturbation δ. □

Proof of Proposition 3

Proof. I begin with the "only if" direction. Suppose that T is coarsely tenable. Suppose, seeking contradiction, there is some sub-block $I \times J$ that possesses a solution to the above systems at $\delta = 0$, but fails at least one of (1) or (2). Fix this sub-block and suppose without loss of generality that (1) fails. Then Ω_{IJ}^2 is nonempty, and there is some $s_\omega \in \Omega_{IJ}^2$ such that S_{IJ}^1 possesses a vertex within the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$, call this vertex (p^*, δ^*). Furthermore, since $s_\omega \in \Omega_{IJ}^2$, the hyperplane $\{(p, \delta) : pBe_\omega - pBe_j\}$ intersects S_{IJ}^1 at a point where $\delta_1 = 0$. Hence there is some point $(\hat{p}, 0) \in S_{IJ}^1$ and satisfies $pBe_\omega = pBe_j$. Using the fact that S_{IJ}^1 is a convex set, it holds that $\lambda(p^*, \delta*) + (1 - \lambda)(\hat{p}, 0) \in S_{IJ}^1$ for all $\lambda \in [0, 1]$. Note also that

$$(\lambda p^* + (1 - \lambda)\hat{p})Be_\omega - (\lambda p^* + (1 - \lambda)\hat{p})Be_j > 0 \ \ \forall \lambda \in (0, 1]$$

Therefore, for any $\lambda \in (0, 1)$, there is a tuple $(\lambda p^* + (1 - \lambda)\hat{p}, \lambda\delta^*, q, 0)$ that solves the above system, and in which s_ω is a strictly better reply to $\lambda p^* + (1 - \lambda)\hat{p}$ than all elements of T_2. Sending $\lambda \longrightarrow 0$ and invoking Lemma 3 yields a contradiction.

Proceeding with the "if" direction, suppose that all sub-blocks outlined in the proposition satisfy (1) and (2). Suppose, seeking contradiction, that T is not coarsely tenable. Then there exists a sequence $\epsilon_k \longrightarrow 0$ and a sequence of (without loss of generality *simple*) distributions μ_k with $\mu_k(T) > 1 - \epsilon_k$ for each k such that there is an equilibrium τ_k of the consideration-set game induced by μ_k where some player, without loss of generality player 2, possesses a strategy $s_\omega \in S_2 \backslash T_2$ such that

$$u_2(s_\omega, \tau_{1,k}^\mu) > max_{t_2 \in T_2} u_2(t_2, \tau_{1,k}^\mu)$$

Consider the sequences $\{\tau_{1|T_1}\}_k$ and $\{\tau_{2|T_2}\}_k$ of conditional distributions over pure strategies in T at equilibrium. The support of $\tau_{i|T_{i}k}$ is a finite subset of T_i for each k. The Bolzano-Weierstrass theorem then implies that the sequence of supports possesses a convergent (constant) subsequence. Hence one can assume that the above sequence is constructed such that the supports of $\{\tau_{i|T_i}\}_k$ remain constant for each i. Denote the support of $\{\tau_{1|T_1}\}_k$ by I, and the support of $\{\tau_{2|T_2}\}_k$ by J. Lemmas 2 and 5 imply that τ_k corresponds to some $(p, \delta_1) \in S_{IJ}^1$ and $(q, \delta_2) \in S_{IJ}^2$ under the chosen sub-block of supports for each k. The fact that the equilibria of the consideration-set games converge to equilibria of the block game G^T as $\mu(T) \longrightarrow 1$ implies that the above sequence possesses a subsequence which converges to an equilibrium τ^* of the block game. I suppose without loss of generality that the entire sequence has this property. This implies that S_{IJ}^1 and S_{IJ}^2 have solutions when $\delta_i = 0$. The fact that

$$u_2(s_\omega, \tau_{1,k}^\mu) > max_{t_2 \in T_2} u_2(t_2, \tau_{1,k}^\mu)$$

for each k implies that the associated solution to the system is such that, for each k,

$$p_k Be_\omega - p_k Be_j > 0$$

where $p_k = \tau^{\mu}_{1,k}$ is the distribution over pure strategies for player 1 induced by τ_k. Furthermore, sending $\epsilon_k \longrightarrow 0$ ($\mu_k(T) \longrightarrow 1$) implies that the hyperplane $\{(p,\delta) : pBe_\omega - pBe_j = 0\}$ intersects the set of solutions to the system at a point in which $\delta_1 = 0$. Therefore, S^1_{IJ} has a vertex that lies within the half-space $\{(p,\delta) : pBe_\omega - pBe_j > 0\}$. This is a contradiction. Thus the result is proved. □

Proof of Proposition 4

Proof. It suffices only to check sub-blocks $I \times J$ in which both S^1_{IJ} and S^2_{IJ} possess solutions on the zero perturbation region. All other sub-blocks will of course lack $I \times J$*-refutations* since solutions of the corresponding system are bounded away from the zero perturbation region.

Suppose that no sub-block $I \times J$ of T possesses an $I \times J$*-refutation*. Suppose, seeking contradiction, that T is not finely tenable. Then there exists a sequence $\{\epsilon_k, \mu_k, \tau_k\}_{k=1}^{\infty}$ such that $\epsilon_k \longrightarrow 0$, μ_k is an $\epsilon_k - proper$ type distribution with respect to T and τ_k is an equilibrium of the consideration-set game induced by μ_k with the property that for each k, there is some player i and strategy $s_\omega \in S_i \backslash T_i$ such that

$$u_i(s_\omega, \tau_{-i,k},) > max_{t_i \in T_i} u_i(t_i, \tau_{-i,k})$$

The sequence can be chosen such that the above inequality holds for the same player and the same strategy s_ω for each k. Furthermore, it is safe to assume that $supp_i(\tau_{i|T_i})_k$ is constant for each player i and all k. These follow from application of Bolzano-Weierstrass. Let $I = supp_1(\tau_{1|T_1})_k$ and $J = supp_2(\tau_{2|T_2})_k$. For each k, Lemma 4 implies that there is a $T(\epsilon_k^{\frac{1}{3}}) - proper$ tuple (σ_k, δ_k) that solves the systems S^1_{IJ} and S^2_{IJ}. This implies that the sequence of tuples $\{\epsilon_k^{\frac{1}{3}}, \sigma_k, \delta_k\}_{k=1}^{\infty}$ obtained from the projections of $\tau_k^{\mu_k}$ constitutes an $I \times J$*-refutation*, a contradiction.

Proceeding with the "only if" direction, suppose that T is finely tenable. Suppose, seeking contradiction, that there is some sub-block $I \times J$ of T which possesses an $I \times J$*-refutation*. Fix this sub-block and let $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ denote a refutation. Suppose without loss of generality that $m = |S_1| = max\{|S_1|, |S_2|\}$. Since $\epsilon_k \longrightarrow 0$, there is $K \in \mathbb{N}$ such that for all $k \geq K$, $\epsilon_k \leq \frac{1}{m^2 2^{2m}} = \epsilon^*$. Proposition 2 then implies that for all $k \geq K$, there is a type distribution μ_k that is $\epsilon^{\frac{1}{2m}} - proper$ and τ_k such that $\tau_k^{\mu_k} = \sigma_k$. This immediately provides a sequence $\{(\epsilon_k)^{\frac{1}{2m}}, \mu_k, \tau_k,\}_{k \geq K}$ that constitutes a direct violation of fine tenability, a contradiction. □

Proof of Lemma 6

Proof. The first condition is clearly necessary, as $T(\epsilon) - proper$ tuples tend to zero perturbation at $\epsilon \longrightarrow 0$.

Suppose, seeking contradiction, that $I \times J$ contains $T(\epsilon) - proper$ tuples (σ_1, δ_1) according to the order $\gtrsim_1$ for all $\epsilon > 0$, but that the second condition above fails. Then there is some strategy $s^* \in S_1$ such that for every vertex v of $cl(S^1_{IJ} \cap p(\gtrsim_2))$ of the form

$$\{v : v(\delta_1(s^*)) > 0\} := Pos(s^*)$$

there is some strategy $s' \in S_1$ in which $s > s'$ and $v(\delta_1(s')) > 0$. Let

$$\epsilon^* = \frac{min_{v \in Pos(s^*)} min_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))}{|Pos(s^*)| max_{\{v \in Pos(s^*)\}} v(\delta_1(s^*))}$$

I claim that there is no tuple $(\sigma_1, \delta_1) \in cl(S^1_{IJ} \cap p(\succsim_2))$ that is $T(\frac{\epsilon^*}{2}) - proper$ according to the order $\succsim_1$. Let,

$$\{v : v(\delta_1(s^*)) = 0\} := Zero(s^*)$$

and consider any element

$$(\sigma_1, \delta_1) = \sum_{v \in Pos(s^*)} \lambda_v v + \sum_{v \in Zero(s^*)} \lambda_v v$$

within $cl(S^1_{IJ} \cap p(\succsim_2))$. In order for properness to hold for any ϵ, it must be that $\delta_1(s) > 0$ for all $s \in S_1$. It must then be that there is $v \in Pos(s^*)$ with $\lambda_v > 0$. Let $\lambda^*_{Pos(s^*)} = max_{\{v \in Pos(s^*)\}} \lambda_v$ and let v^* be a maximizing vertex. Choose any $s' \prec_1 s$ such that $v^*(\delta_1(s')) > 0$. Then,

$$\frac{\delta_1(s')}{\delta_1(s^*)} = \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s')) + \sum_{v \in Zero(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))} \geq \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))}$$

$$\geq \frac{\lambda^*_{Pos(s^*)} min_{v \in Pos(s^*)} min_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))}{\lambda^*_{Pos(s^*)} |Pos(s^*)| max_{\{v \in Pos(s^*)\}} v(\delta_1(s^*))} = \epsilon^*$$

Hence there are no $T(\frac{\epsilon^*}{2}) - proper$ tuples in $cl(S^1_{IJ} \cap p(\succsim_2))$, a contradiction. □

Proof of Lemma 7

Proof. The proof is a straightforward result of the fact that S^1_{IJ} and $p(\succsim_2)$ are convex. This implies that their intersection is convex, and hence

$$relint(cl(S^1_{IJ}) \cap p(\succsim_2)) = relint(S^1_{IJ} \cap p(\succsim_2)) \subseteq S^1_{IJ} \cap p(\succsim_2)$$

where the first equality follows from known results about convex sets. (See Proposition 1.4.3 of [Bertsekas et al. 2003]).

□

Proof of Lemma 8

Proof. Given $\epsilon \in (0, 1)$, I seek to construct a tuple (σ_1, δ_1) that satisfies the properness criteria for ϵ and such that

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))\}} \lambda_v v$$

with $\lambda_v > 0$ for all vertices v. Fix the following values:

$$\bar{\delta} = max_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))} \; max_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))$$

$$\underline{\delta} = min_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))} \; min_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))$$

Partition S^1 into the indifference classes $\zeta^1_1, \zeta^1_2, ..., \zeta^1_{r_1}$ according to $\succsim_1$, but in decreasing order this time (i.e. ζ^1_1 contains the best strategies according to $\succsim_1$). For each ζ^1_k and each $s \in \zeta^1_k$, the conditions of Lemma 6 imply that there is a corresponding vertex v_s such that $v_s(\delta_1(s)) > 0$ and for each ζ^1_j with $j > k$,

$$v_s(\delta_1(s')) = 0 \;\; \forall s' \in \zeta^1_j$$

Note that multiple strategies within the same indifference class may possess the same corresponding vertex. For each indifference class ζ_k^1, choose a collection of vertices V_k such that for each strategy $s \in \zeta_k^1$, there is a vertex $v_s \in V_k$ such that $v(\delta_1(s)) > 0$ and for each ζ_j^1 with $j > k$,

$$v_s(\delta_1(s')) = 0 \ \ \forall s' \in \zeta_j^1$$

Note that $V_k \cap V_j = \emptyset$ for $k \neq j$. For $v \in V_k$, let

$$\lambda_v = \frac{(\epsilon\underline{\delta})^k}{(|S^1|d)^k}$$

where d is the number of vertices of $cl(S_{IJ}^1 \cap p(\succsim_2))$. For all vertices $v \notin \bigcup_k V_k$ that do not lie on the zero perturbation region, let

$$\lambda_v = \frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}}$$

For vertices on the zero perturbation region (of which there is at least one by assumption), allocate all remaining probability uniformly. Note that under this construction all λ_v can be scaled down in order to construct a valid probability distribution while maintaining the requisite properness criteria. Let us now verify that the profile

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S_{IJ}^1 \cap p(\succsim_2))\}} \lambda_v v$$

is $T(\epsilon) - proper$. The fact that $(\sigma_1, \delta_1) \in S_{IJ}^1$ (Lemma 7) implies that the first two conditions are satisfied. The third condition is clearly satisfied. To verify the fourth condition, let $s \in \zeta_k^1$ and $s' \in \zeta_{k+1}^1$ for $k \leq r_1 - 1$. Then

$$\delta_1(s) \geq \frac{(\epsilon\underline{\delta})^k}{(|S^1|d)^k}$$

$$\delta_1(s') \leq \bar{\delta}(d - \sum_{i=1}^{r_1} |V_i|)\frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon\underline{\delta})^i \bar{\delta}}{(|S^1|d)^i}$$

then

$$\frac{\delta_1(s')}{\delta_1(s)} \leq \frac{\bar{\delta}(d - \sum_{i=1}^{r_1} |V_i|)\frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon\underline{\delta})^i \bar{\delta}}{(|S^1|d)^i}}{\frac{(\epsilon\underline{\delta})^k}{(|S^1|d)^k}}$$

$$\leq \frac{d\bar{\delta}(\epsilon\underline{\delta})^{k+1}}{(\epsilon\underline{\delta})^k} \frac{(|S^1|d)^k}{(|S^1|d)^{k+1}} = \frac{\bar{\delta}\underline{\delta}\epsilon}{|S^1|} \leq \epsilon$$

The final condition for (σ_1, δ_1) to be $T(\epsilon) - proper$ can be made to hold due to the fact that all λ_v that do not lie on the zero perturbation region may be scaled down while maintaining the first four conditions. Hence it is possible to extract a profile $(\sigma_1, \delta_1) \in relint(cl(S_{IJ}^1) \cap p(\succsim_2))$ which is $T(\epsilon) - proper$ when the conditions of Lemma 6 are satisfied. □

Proof of Proposition 5

Proof. Suppose that $I \times J$ possesses a refutation $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ with constant total preorders $\{\succsim_1, \succsim_2\}$. One must show that

(1) The vertices of $cl(S_{IJ}^1 \cap p(\succsim_2))$ and $cl(S_{IJ}^2 \cap q(\succsim_1))$ are also vertices of H_{IJ}^1 and H_{IJ}^2 respectively.
(2) The total preorders $\succsim_1$ and $\succsim_2$ satisfy the restrictions imposed by steps $2 - 5$ of Algorithm 1.
(3) The total preorders $\succsim_1$ and $\succsim_2$ satisfy the conditions of Lemma 6.

To verify (1), note that $cl(S^1_{IJ} \cap p(\succsim_2))$ corresponds to a face of the arrangement H^1_{IJ}. Similarly $cl(S^2_{IJ} \cap q(\succsim_1))$ is a face of H^2_{IJ}.
Condition (2) must clearly be satisfied as a refutation profile σ_k with constant total preorders for all k has a subsequence which converges to some point on $Z^1_{IJ} \times Z^2_{IJ} = Conv(V^1_Z) \times Conv(V^2_Z)$ as $\epsilon_k \longrightarrow 0$.
Lemmas 6-8 guarantee that if such (σ_k, δ_k) exists then the sets $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ satisfy the conditions of Lemma 6. Thus Algorithm 1 detects the refutation $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ provided it compares the preorders $\{\succsim_1, \succsim_2\}$ associated with the sequence σ_k.

On the other hand, if the algorithm encounters two total preorders $\succsim_1$ and $\succsim_2$ such that $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ satisfy the conditions of Lemma 6, then Lemma 6 implies that it is possible to construct an *I* × *J-refutation.* □

Proof of Lemma 9

Proof. Begin by proving the first claim of the lemma. Suppose seeking contradiction and w.l.o.g. that player one possesses a pair of variables x_j and x_k such that

$$\sigma_1(+l_j) + \sigma_1(-l_j) > \sigma_1(+l_k) + \sigma_1(-l_k)$$

Then it must be that

$$u_2(\sigma_1, x_j) > u_2(\sigma_1, x_k)$$

Thus $x_j \succ_2 x_k$ and $\sigma_2(x_j)\epsilon \geq \sigma_2(x_k)$. Note that as $\epsilon \longrightarrow 0$ one has that

$$x \succ_2 l \quad \forall x \in V, l \in L$$

and so

$$\sigma_2(x_j)\epsilon^2 \geq \sigma_2(x_k)\epsilon \geq \sigma_2(l) \quad \forall l \in L$$

As $\epsilon \longrightarrow 0$, it must then be that

$$u_1(\pm l_j, \sigma_2) < u_1(\pm l_k, \sigma_2)$$

and thus σ_1 does not satisfy criteria for properness.
Similarly, suppose that $\sigma_1(x_j) > \sigma_1(x_k)$. Then $\pm l_k \succ_2 \pm l_j$, and hence player two places much less probability on $\{+l_j, -l_j\}$ than on $\{+l_k, -l_k\}$ as $\epsilon \longrightarrow 0$. This implies that $u_1(x_j, \sigma_2) < u_1(x_k, \sigma_2)$, and hence σ_1 cannot be proper. □

Proof of Lemma 10

Proof. Suppose without loss of generality and seeking contradiction that σ_1 is such that there exists two variables x_i and x_j with $max\{\sigma_1(+l_i), \sigma_1(-l_i)\} > max\{\sigma_1(+l_j), \sigma_1(-l_j)\}$. Lemma 9 of course implies that $min\{\sigma_1(+l_i), \sigma_1(-l_i)\} < min\{\sigma_1(+l_j), \sigma_1(-l_j)\}$. This implies that

$$max\{u_2(\sigma_1, +l_i), u_2(\sigma_1, -l_i)\} > max\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\} \geq min\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\}$$

and hence $\sigma_2(+l_i) + \sigma_2(-l_i) > \sigma_2(+l_j) + \sigma_2(-l_j)$ in order to maintain properness. But this violates Lemma 9, a contradiction. Thus for every pair of variables x_i and x_j

$$max\{\sigma_i(+l_i), \sigma_i(-l_i)\} = max\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

$$min\{\sigma_i(+l_i), \sigma_i(-l_i)\} = min\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

for each player *i*. Thus a partition L^i_1 and L^i_2 exists for individual players. To show that $L^1_1 = L^2_1$, note that if $\sigma_i(+l_j) > \sigma_i(-l_j)$ then $u_{-i}(\sigma_{-i}, +l_j) > u_{-i}(\sigma_{-i}, -l_j)$ and hence $\sigma_{-i}(+l_j) > \sigma_{-i}(-l_j)$. Similarly it is easy to prove that if $\sigma_i(+l_j) = \sigma_i(-l_j)$, it must be that $\sigma_{-i}(+l_j) = \sigma_{-i}(-l_j)$. Thus it must be that $L_1^1 = L_1^2$ and $L_2^1 = L_2^2$. □

Proof of Lemma 11

Proof. Let $A = \{l_1, l_2, ..., l_n\}$ denote a satisfying assignment of ϕ. Let $C_1 = \{c \in C : c \cap A = 1\}$, $C_2 = \{c \in C : c \cap A = 2\}$ and $C_3 = \{c \in C : c \cap A = 3\}$. Construct a refutation as follows:

$$\delta_i(f) = \sigma_i(f) = \epsilon$$
$$\delta_i(c) = \sigma_i(c) = \epsilon^2 \ \ \forall c \in C_1$$
$$\delta_i(c) = \sigma_i(c) = \epsilon^3 \ \ \forall c \in C_2$$
$$\delta_i(c) = \sigma_i(c) = \epsilon^4 \ \ \forall c \in C_3$$
$$\delta_i(g) = \epsilon^5$$
$$\delta_i(x) = \sigma_i(x) = \epsilon^6 \ \ \forall x \in V$$
$$\delta_i(l) = \sigma_i(l) = \epsilon^7 \ \ \forall l \in A$$
$$\delta_i(h) = \sigma_i(h) = \epsilon^8$$
$$\delta_i(l) = \sigma_i(l) = \epsilon^9 \ \ \forall l \notin A$$
$$\sigma_i(g) = 1 - \sum_{s \neq g} \sigma_i(s)$$

I have omitted the normalizing factor that would ensure that $\sigma_i(g) > 1 - \epsilon$, but this does not present an issue to the proof. All probabilities aside from $\sigma_i(g)$ could be scaled down by for instance a factor of $\frac{1}{2}$ to ensure this holds. Now to show that it is indeed the case that

$$f \succ_i c \succ_i g \succ x \succ_i l(\in A) \succ_i h \succ_i l(\notin A)$$

for each player i. Note that since $\sigma_i(g) \rightarrow 1$ as $\epsilon \rightarrow 0$, it must be that $f, g, c \succ x \succ l, h$ for small ϵ. Computing expected payoffs:

$$u_i(\sigma_{-i}, f) = (\frac{n - \frac{1}{2}}{n})(n\epsilon^7 + n\epsilon^9) + (1 - n\epsilon^7 - n\epsilon^9)(n-1)$$
$$u_i(\sigma_{-i}, c) = (n-1)\epsilon^7 + (n-2)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \ \ \forall c \in C_1$$
$$u_i(\sigma_{-i}, c) = (n-2)\epsilon^7 + (n-1)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \ \ \forall c \in C_2$$
$$u_i(\sigma_{-i}, c) = (n-3)\epsilon^7 + (n)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \ \ \forall c \in C_3$$
$$u_i(\sigma_{-i}, g) = n(|C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4) + (1 - \epsilon - |C_1|\epsilon^2 - |C_2|\epsilon^3 - |C_3|\epsilon^4)(n-1)$$
$$u_i(\sigma_{-i}, l) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + n(n-1)\epsilon^7 + (n-1)^2\epsilon^9 + (n-4)\epsilon^9 + (n-3)\sigma_i(g) \ \ \forall l \in A$$
$$u_i(\sigma_{-i}, h) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + n(n - 1 - \frac{1}{2n})(\epsilon^7 + \epsilon^9) + (n-3)\sigma_i(g)$$
$$u_i(\sigma_{-i}, l) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 + (n-1)^2\epsilon^7 + (n-4)\epsilon^7 + n(n-1)\epsilon^9 + (n-3)\sigma_i(g) \ \ \forall l \notin A$$

Using these calculated payoffs, it is possible to verify that the appropriate ordering holds for $\succsim_i$, and hence this profile serves as a refutation.

□

Proof of Lemma 12

Proof. In this case, I will show that at every $T(\epsilon) - proper$ profile, with ϵ small, there is $c^* \in C$ such that $c^* \succ_i f$ for each player i, and hence it must be that $g \succ_i c^* \succ_i f$. This would imply that no refutations exist. Let L^1 and L^2 be any partition of literals L from Lemma 10. There are two relevant cases to consider.

- Case 1: $l_1 \succ_i l_2 \;\; \forall l_1 \in L_1, l_2 \in L_2$ and for each player i.
  In this case, fix L_1 and L_2, and consider the ordering $\succ_i$ for each player i. Evidently it must be that $l(\in L_1) \succ_i h \succ_i l(\in L_2)$ for each player i. Furthermore, since L_1 does not form a satisfying assignement to ϕ, there is some $c^* \in C$ such that $L_1 \cap C = \emptyset$. Hence $c^* \succ_i f$ for each player i, and consequently $g \succ_i c^* \succsim_i s \;\; \forall s \neq g, c^*$.
- Case 2: $l_1 \sim_i l_2 \;\; \forall l_1 \in L_1, l_2 \in L_2$ and for each player i.
  In this case, one has that $h \succ_i l \quad \forall l \in L$ for each player i. Indeed, this was the purpose of including h in the construction: to block uniform randomization over all literals as a refutation. Since $h \succ_i l \;\; \forall l \in L$, it must be that $c \succ_i f \;\; \forall c \in C$. It immediately follows that $g \succ_i c \;\; \forall c \in C$, and consequently g is a (strict) best response for each player.

□

Proof of Lemma 13

Proof. Suppose that one is given a sub-block $I \times J$ of T and a pair of total preorders $\{\succsim_1, \succsim_2\}$. To verify whether they serve as a refutation, it suffices to show that $p(\succsim_2)$ contains profiles which are $T(\epsilon)$*-proper* in its relative interior, with a similar statement holding for $q(\succsim_1)$ with ϵ sufficiently small. Lemmas 6-8 imply that one need only determine whether there are a collection of vertices of $p(\succsim_2)$ and $q(\succsim_1)$ that satisfy the conditions of lemma 6. Consider w.l.o.g. the total preorder $\succsim_1$ and enumerate the indifference classes $\zeta_1^1, ..., \zeta_r^1$ in decreasing order, so that strategies in ζ_i^1 are strictly preferred to all $s' \in \zeta_j^1$ with $j > i$. To verify the conditions of lemma 6 for a given strategy $s \in \zeta_i^1$, it suffices to solve

$$\max \delta_1(s)$$

$$s.t. \;\; \delta_1(s') = 0 \;\; \forall s' \in \zeta_j^1, j > i$$

$$\delta_1(s) \in cl(S_{IJ}^1 \cap p(\succsim_2))$$

and verify whether there is a basic feasible solution with $\delta_1(s) > 0$. Note that the constraint set for this problem is defined by a number of linear inequalities that is polynomial in $|S_1|$ and $|S_2|$. If the desired solution exists for all $s \in S_1$, then lemma 8 implies that there are $T(\epsilon)$*-proper* profiles according to $\succsim_1$ that lie within $p(\succsim_2)$. A similar process can be performed for strategies in $\succsim_2$. □

Proof of Proposition 8

Proof. By proposition 4, the problem of verifying whether a given block T is finely tenable is equivalent to determining whether there is a sub-block $I \times J$ that possesses an $I \times J$*- refutation*. Lemmas 6-8 imply that any $I \times J$*refutation* can be equivalently expressed as a pair of total preorders $\{\succsim_1, \succsim_2\}$. Thus the problem of verifying fine tenability of a block T is equivalent to finding $\{I, J, \succsim_1, \succsim_2\}$ with $I \times J$ a sub-block of T and $\succsim_1, \succsim_2$ satisfying the conditions outlined in lemma 6. Given such a tuple $\{I, J, \succsim_1, \succsim_2\}$, lemma 14 implies that it can be verified in polynomial time. This, combined with proposition 7, implies that the problem is NP-complete. □

Proof of Lemma 14

Proof. To prove necessity, suppose that $\sigma = (p^*, q^*)$ is indeed a proper equilibrium. Then there is a sequence $\{\epsilon_k, \sigma_k\}_{k=1}^{\infty}$ of ϵ_k-*proper* equilibria converging to σ with $\epsilon_k \longrightarrow 0$. The total preorders over strategies induced by $\sigma_{1,k}$ and $\sigma_{2,k}$ can be taken to be constant in k. Hence the sequence $\sigma_{1,k}$ and $\sigma_{2,k}$ lie on single closed faces f_1 and f_2 respectively. Since these faces are closed and the sequences converge to p^* and q^*, it must be that $p^* \in f_1$ and $q^* \in f_2$. The fact that σ_k are ϵ_k-*proper* with $\epsilon_k \longrightarrow 0$ implies that $f_i \subseteq P^i(p^*, q^*)$. Showing the final condition is almost identical to the logic of Lemma 6.

To show sufficiency, it suffices to construct a convergent sequence of ϵ_k-*proper* equilibria directly. Since $p^* \in f_1$, it follows that for every strategy $s \in Y_1(p^*)$, there is a vertex v of f_1 in which $v(p_s) > 0$ and in which $v(p_{s'}) = 0$ for all $s' \notin Y_1(p^*)$. Furthermore, p^* can be realized as a convex combination of such vertices. Let V_{p^*} denote the largest collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in Y_1(p^*)$ and $v_(p_s') = 0$ for all $s' \notin Y_1(p^*)$, and let $\{\lambda_v^*\}_{v \in V_{p^*}}$ denote a convex combination such that

$$\sum_{v \in V_{p^*}} \lambda_v^* v = p^*$$

Let $Z(V_{p^*}) = \{v \in V_{p^*} : \lambda_v^* = 0\}$. Fix the following values:

$$\bar{p} = max_{v \in f_1} max_{\{s : v(p_s) > 0\}} v(p_s)$$

$$\underline{p} = min_{v \in f_1} min_{\{s : v(p_s) > 0)\}} v(p_s)$$

Let $\gtrsim_1$ denote the total preorder induced over S_1 induced by strategies on the relative interior of f_2. Let V_1 denote the collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in \zeta_1^1 \backslash Y_1(p^*)$ and $v(p_{s'}) = 0$ for all $s' \in \zeta_k^1$, $k > 1$. Similarly, let V_k, $k \geq 2$, denote the collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in \zeta_k^1$ and $v(p_{s'}) = 0$ for all $s' \in \zeta_l^1$, with $l > k$. Note that $V_k \cap V_j = \emptyset$ for $j \neq k$. Let $|V(f_1)|$ denote the total number of vertices of the closed face f_1. For a given, small $\epsilon > 0$, I will construct a profile p that lies within the relative interior of f_1 and is ϵ-*proper* according to $\gtrsim_1$. For $v \in V_k$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^k}{(|S_1||V(f_1)|)^k}$$

For $v \notin \left\{ \cup_k V_k \right\} \cup V_{p^*}$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

For $v \in Z(V_{p^*})$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

For $v \in V_{p^*} \backslash Z(V_{p^*})$, define

$$\lambda_v = (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda_{v'}) \lambda_v^*$$

Note that as $\epsilon \longrightarrow 0$,

$$\sum_{v' \notin V_{p^*} \backslash Z(V_{p^*})} \lambda_{v'} \longrightarrow 0$$

Therefore

$$p_\epsilon = \sum_{v \in f_1} \lambda_v v \longrightarrow p^*$$

as $\epsilon \longrightarrow 0$ since $p^* = \sum_{v \in V_{p^*}} \lambda_v^* v$. Note also that $p_\epsilon(s) > 0$ for all $s \in S_1$ and all $\epsilon > 0$. All that is left to show is that p_ϵ is indeed ϵ-*proper* according to $\gtrsim_1$ for small enough ϵ. For any $s \in \zeta_k^1 \backslash Y_1(p^*)$, with $1 \leq k \leq r_1 - 1$,

$$p_\epsilon(s) \geq \frac{(\epsilon \underline{p})^k}{(|S^1||V(f_1)|)^k}$$

and for any $s' \in \zeta_{k+1}^1$

$$p_\epsilon(s') \leq \sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

and hence

$$\frac{p_\epsilon(s')}{p_\epsilon(s)} \leq \frac{\sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}}{\frac{(\epsilon \underline{p})^k}{(|S^1||V(f_1)|)^k}}$$

$$\leq \frac{|V(f_1)| \bar{p} (\epsilon \underline{p})^{k+1}}{(|S_1||V(f_1)|)^{k+1}} \frac{(|S_1||V(f_1)|)^k}{(\epsilon \underline{p})^k} = \frac{\bar{p}(\epsilon \underline{p})}{|S_1|} \leq \epsilon$$

Now let $s \in Y_1(p^*)$ and $s' \in \zeta_2^1$. Then

$$p(s) \geq \underline{p} min_{\{v : \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda'_v)$$

$$p(s') \geq \sum_{i=2}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

Then

$$\frac{p(s')}{p(s)} \leq \frac{\bar{p} \underline{p} \epsilon^2}{|S_1|^2 |V(f_1)| min_{\{v : \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda'_v)} \leq \epsilon$$

for ϵ small enough. Thus p_ϵ constructed here is a sequence of ϵ-*proper* equilibria which converge to p^* as $\epsilon \longrightarrow 0$. Note that since p_ϵ are all in the relative interior of f_1, they induce the same total preorder over S_2. A mirrored construction allows one to construct q_ϵ which are ϵ-*proper* for ϵ small enough and which lies in the relative interior of f_2, and hence induce the same total preorder over strategies in S_1. This provides a pair (p_ϵ, q_ϵ) satisfying the desired properties for ϵ small and converging to (p^*, q^*) as $\epsilon \longrightarrow 0$. □